

# STABLE CONNECTIVITY OVER A BASE

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ABSTRACT. The Morel's stable connectivity theorems [Mor1] states that for an arbitrary field for any connective  $S^1$ -spectrum  $F$  of Nisnevich sheaves, the spectrum  $L_{\mathbb{A}^1}(F)$  is connective, and the same property for  $\mathbb{P}^1$ -spectra of Nisnevich sheaves. Here  $L_{\mathbb{A}^1}$  denotes the  $\mathbb{A}^1$ -Nisnevich-localisation in the category  $\mathbf{SH}_{\text{nis}}^{S^1}(k)$  of  $S^1$ -spectra of Nisnevich sheaves over a field  $k$ . In view of the Ayoub's contrexamples to the relative case of a  $S^1$ -spectra sheaves over a base scheme  $S$  [Ayo06] the conjecture states that  $L_{\mathbb{A}^1}(F)$  is  $(-d)$ -connective for any connective  $F$ , where  $d = \dim S$  is the Krull dimension. The conjecture is proven for the cases of dedekind schemes in [SS] and noetherian domains in [DHKY] under the assumption of the infiniteness of the residue fields. In the article we prove the general case.

## 1. INTRODUCTION

1.1. **The stable connectivity theorem in  $\mathbf{SH}(S)$  and  $\mathbf{SH}_{S^1}(S)$ .** Following the original definitions [MV99] a pointed motivic space  $F$  over a base scheme  $S$  is a Nisnevich simplicial sheave on the category of smooth schemes over  $S$ , and a motivic  $S^1$ -spectrum is an  $S^1$ -spectrum of pointed motivic spaces. Denote by  $\mathbf{SSh}^\bullet(S)$  the category of pointed motivic spaces over  $S$ , and denote by  $\mathbf{SSh}^{S^1}(S)$  the category of  $S^1$ -spectra of pointed motivic spaces. Equivalently we could say that  $\mathbf{SSh}^{S^1}(S)$  is the category of Nisnevich sheaves of  $S^1$ -spectra.

According to [Mor1] An  $S^1$ -spectrum  $E$  is called  $n$ -connective, iff all the negative stable homotopy groups of  $E \wedge S^n$  are trivial, and an  $S^1$ -spectrum of pointed motivic spaces  $\mathcal{F} = (F_0, F_1, \dots, F_i, \dots)$  is called  $n$ -connective, iff  $\forall U \in Sm_S$  the  $S^1$ -spectrum  $\mathcal{F}(U) = (F_0(U), F_1(U), \dots, F_i(U), \dots)$  is  $n$ -connective. Precisely, this means that  $\forall U \in Sm_S$  the stable homotopy groups

$$\pi_i(\mathcal{F}(U)) = 0, \forall i < n,$$

where

$$\pi_i(\mathcal{F}(U)) = [S^0, \mathcal{F}(U) \wedge S^i]_{\mathbf{SH}} = \varinjlim [S^j, \mathcal{F}(U) \wedge S^{i+j}]_{\mathbf{H}},$$

and  $\mathbf{SH}$  and  $\mathbf{H}^\bullet$  denotes the (classical topological) stable homotopy category and pointed unstable homotopy category.

The Morel's stable connectivity theorem

**Theorem 1** ( Theorem 6.1.8 [Mor1]). *For an arbitrary base field  $k$  if an  $S^1$ -spectrum of Nisnevich sheaves  $\mathcal{F} \in \mathbf{SSh}^{S^1}(k)$  is 0-connective, then  $L_{\mathbb{A}^1}(\mathcal{F})$  is 0-connective. where  $L_{\mathbb{A}^1} : \mathbf{SSh}_{\text{nis}}^{S^1}(k) \rightarrow \mathbf{SSh}_{\text{nis}}^{S^1}(k)$  is the  $\mathbb{A}^1$ -localisation functor, that is the localisation functor with respect to the Bousfield localisation on  $\mathbf{SSh}^{S^1}$  generated by the equivalences of the form  $U \times \mathbb{A}^1 \rightarrow U$ .*

The original Morel's stable connectivity conjecture claims that the same property holds in  $\mathbf{SH}_{S^1}(S)$  for an arbitrary base scheme  $S$ . In [Ayo06] it was shown by Ayoub that there are such contr-examples that are given by pointed motivic  $S^1$ -spectra  $F$  such that  $L_{\mathbb{A}^1}(F)$  is not 0-connective, but only  $(-d)$ -connective, where  $d = \dim S$ . The modified version of Morel's conjecture in view of Ayoub's contr-examples is the following statement for a base scheme  $S$ :

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**Conjecture 1.** *Let  $S$  be a base scheme,  $\dim S = d$ . If an  $S^1$ -spectrum of Nisnevich sheaves  $\mathcal{F} \in \mathbf{Sh}_{\text{nis}}^{S^1}(S)$  is 0-connective, then the spectrum  $L_{\mathbb{A}^1}(\mathcal{F})$  is  $(-d)$ -connective, where  $\mathbf{Sh}_{\text{nis}}^{S^1}(S)$  denotes the category of  $S^1$ -spectrum of Nisnevich sheaves over  $S$ .*

The connectivity theorem in such form was proven in [SS] by J. Schmidt and F. Strunk for the case of dedekind base schemes with infinite residue fields. In [DHKY] the result is extended by N. Deshmukh, A. Hogadi, G. Kulkarni and S. Yadav to the case of an arbitrary noetherian domains with an infinite residue fields. In the article we prove the result for an arbitrary base scheme of a finite Krull dimension with an arbitrary residue fields.

In [SS] and [DHKY] the connectivity theorem is proven by using of the original scheme over the reasoning and proving of the corresponding Gabber's presentation lemma or the corresponding cases of the base scheme. In the present work we don't use and prove the Gabber's presentation lemma (in general), but we follow the same kind of arguments as in the original reasoning [Mor1] to prove the connectivity theorem.

**1.2. The proof of stable connectivity over a field.** The proof of the stable connectivity theorem over a field  $k$  consists in short of two steps:

**Theorem** (Lemma 4.1.6, [Mor1]). *Let  $X$  a smooth scheme over a field  $k$ . Let  $Z \subset X$  be a closed subscheme of positive codimension. Then the motivic space  $X/(X - Z)$  is 0-connective.*

**Theorem** (Theorem 4.14 [2], Lemma 4.3.1 [Mor1], Proposition 3.1 [SS]). *Let  $U \in \text{Sm}_S$  over a base scheme  $S$ , and  $E$  be a connective  $S^1$ -spectrum of Nisnevich sheaves. Then*

$$(1) \quad [U, E \wedge S^i]_{\mathbf{SH}_{S^1}(S)} = 0, \forall i > \dim U.$$

The first theorem is equivalent to the the *injectivity* of the morphism of pointed sets

$$(2) \quad [U, F]_{\mathbf{H}(S)} \hookrightarrow [U - Z, F]_{\mathbf{H}(S)}, \forall U \in \text{EssSm}_S \text{ local henselian, and } \forall F \in \mathbf{H}(S).$$

The connectivity theorem follows, since if  $\eta \in U$  is the generic point, then  $\forall i > 0$

$$\begin{aligned} [U, Y \wedge S^i]_{\mathbf{SH}_{S^1}(k)} &\stackrel{(2)}{\hookrightarrow} [\eta, Y \wedge S^i]_{\mathbf{SH}_{S^1}(k)} \stackrel{(1)}{=} 0, \forall i > 0, \text{ and} \\ [U, Y \wedge S^i]_{\mathbf{SH}(k)} &= \varinjlim_n [\Sigma_{\mathbb{P}^1}^n U, (\Sigma_{\mathbb{G}_m}^n Y) \wedge S^{i+n}]_{\mathbf{SH}_{S^1}(k)} \stackrel{(2)}{\hookrightarrow} [\eta \wedge (\mathbb{P}^1)^{\wedge n}, (\Sigma_{\mathbb{G}_m}^n Y) \wedge S^{i+n}]_{\mathbf{SH}_{S^1}(k)} \stackrel{(1)}{=} 0, \end{aligned}$$

since  $\dim \eta = 0$ , and  $\dim \mathbb{P}_\eta^1 = 1$ ,  $\dim \infty_\eta = 0$ , where  $\infty_\eta$  is the infinity point.

In general relative case over a base scheme  $S$  the vanishing theorem (1) holds, and the injectivity (2) fails.

**1.3. The vanishing theorem (over a base).** The original argument for *vanishing theorem* (1) in [Mor1] is based on the "geometric" model for the stable  $\mathbb{A}^1$ -localisation functor  $L_{\mathbb{A}^1}^{\text{nis}, S^1}$  on the category  $\mathbf{SH}_{\text{nis}}^{S^1}(S)$  of  $S^1$ -spectra of Nisnevich sheaves, that leads to the isomorphism

$$[U, E]_{\mathbf{SH}^{S^1}(S)} \stackrel{\text{def}}{=} [U, L_{\mathbb{A}^1}^{\text{nis}, S^1} E]_{\mathbf{SH}_{\text{nis}}^{S^1}(S)} \simeq \varinjlim_n [U \wedge (\mathbb{A}^1 / \{0, 1\})^n, E]_{\mathbf{SH}_{\text{nis}}^{S^1}(S)}.$$

Then form the case of  $E = \text{EM}(M)$  being Eilenberg-MacLane spectrum of an abelian group Nisnevich sheaf  $M$ , the vanishing (1) follows because of vanishing of Nisnevich cohomologies  $H_{\text{nis}}^i(X, M) = 0$  for all  $i > \dim X$ . Then analyse of the Postnikov tower leads to

$$[U, E \wedge S^i]_{\mathbf{SH}_{\text{nis}}^{S^1}(S)} = 0, \forall U \in \text{Sch}_S, i > \dim U, E \in \mathbf{SH}_{\text{nis}, \geq 0}^{S^1}(\text{Sm}_S)$$

and implies (1) in the case of general connective spectrum of sheaves  $E$ . The complete analyse of the Postnikov towers and the proof of the vanishing theorem in the general relative case is given in [SS, Section 3].

**1.4. Injectivity over a field.** The original proof of *Injectivity* (2) is given by the independent arguments for the perfect base field case [Mor0] and the infinite base field case [Mor1].

In the perfect case for any closed  $Z \subset U$  there is a filtration  $Z_0 \subset \cdots \subset Z_{\dim Z} = Z$  with smooth  $Z_{i+1} - Z_i$ . This allows to reduce the question to the case of smooth  $Z$  that follows by purity isomorphism  $U/(U - Z) \simeq Th(T_{Z/U})$ .

The argument for the infinite field case is based on base on the Gabber's presentation lemma. The Gabber's presentation lemma over finite fields was proven later by A. Hogadi and G. Kulkarni in [HK], by this the second argument had covered the case of an arbitrary base field.

Let us point that also one universal proof of the injectivity theorem (2) in  $\mathbf{H}(k)$  over an arbitrary field follows from the constructions obtained in Panin's work [1] on the Grothendieck-Serre conjecture.

### 1.5. The proof of the connectivity theorem and the injectivity theorem over a base.

**Theorem** (Injectivity over a base). *The injectivity (2) in the case of a local (henselian) base scheme  $S$ , with a closed point  $\sigma \in S$ , holds for an essentially smooth local henselian  $U$  and a closed subset  $Z \subset U$  such that*

$$(3) \quad \text{codim}_{U_\Sigma} Z_\sigma > 0, U_\sigma = U \times_S \sigma, Z_\sigma = Z \times_S \sigma, \sigma \in S$$

*the closed fibre  $Z_\sigma$  is of a positive codimension in the closed fibre  $U_\sigma$ .*

Then connectivity theorem (1) for the case of a local  $S$  with a closed point  $\sigma \in S$  follows, since

$$[U, Y^\wedge S^i]_{\mathbf{SH}_{\mathfrak{s}_1}(S)} \xrightarrow{(2)} [U_\eta, Y^\wedge S^i]_{\mathbf{SH}_{\mathfrak{s}_1}(S)} \stackrel{(1)}{=} 0, \forall i > \dim U, \text{ where } \eta \in U \text{ is the generic point,}$$

since  $\dim U_\eta = \text{codim}_U U_\sigma = \text{codim}_S \sigma \leq \dim S$ . The general case of a base scheme  $S$  follows since for a local henselian  $U$  we have  $[U, Y^\wedge S^i]_{\mathbf{SH}_{\mathfrak{s}_1}(S)} = [U, (Y \times_S S')^\wedge S^i]_{\mathbf{SH}_{\mathfrak{s}_1}(S')}$  where  $S'$  is the local scheme of  $S$  at the point that is the image of the closed point of  $U$ .

## 2. INJECTIVITY THEOREM

**Theorem 2.** *Let  $S$  be a local henselian scheme, and  $\sigma \in S$  be the closed point. Let  $U$  be an essentially smooth local henselian scheme over  $S$ . Let  $Z \subset U$  be a closed subscheme such that the closed fibre  $Z \times_S \sigma$  is a closed subscheme of positive codimension in  $U \times_S \sigma$ .*

*Then for any functor  $F: \mathbf{H}(S) \rightarrow \mathbf{SSet}$  the induced homomorphism  $F(U) \rightarrow F(U/Z)$  is injective.*

*Remark 1.* Equivalently the statement of the theorem means that there is a smooth scheme (of a finite type)  $X$ , with a point  $x \in X$ , and a closed subscheme  $W \subset X$  such that  $U$  and  $Z$  are equal to the local schemes of  $X$  and  $W$  at  $x$ , and the class of the canonical morphism  $U \rightarrow X/(X - W)$  in  $[U_+, X/(X - W)]_{\mathbf{H}_\bullet(S)}$  is equal to the pointed morphism.

*Proof.* Let  $x \in U$  be the closed point. Now let us point that since  $U$  is local henselian, it follows that  $Z$  is either empty either local henselian scheme with the same closed point  $x$ . The case of empty  $Z$  is trivial, so we can assume that  $Z \neq \emptyset$ .

Firstly, we reduce the question to the case of a scheme  $U$  of a relative dimension one over  $S$ . Since  $Z \times_S \sigma$  is of positive codimension in  $U \times_S \sigma$ , and since  $U$  is essentially smooth, by lemma 2 there is a map  $p: U \rightarrow \mathbb{A}_S^{n-1}$  where  $n = \dim_S U$ , such that  $p$  is (essentially) smooth, and  $Z \times_{\mathbb{A}_S^{n-1}} p(x)$  is a closed subscheme of positive codimension in  $\mathbb{A}_S^n$ . So we can redefined  $S$  as the henselization of  $\mathbb{A}_S^n$  at  $p(x)$  and assume by this that  $U$  is a essentially smooth local henselian scheme over  $S$ .

Since  $U$  is or relative dimension one over  $S$ , and  $Z \times_S \sigma$  is of positive codimension in  $U \times_S \sigma$ , it follows that  $Z$  is quasi-finite over  $S$ . Then since  $S$  is local henselian, and  $Z$  is local by the above, it follows that  $Z$  is finite over  $S$ .

By lemma 1 there is an etale map  $\pi: U \rightarrow \mathbb{A}_S^1$  that induces isomorphism  $x \simeq \pi(x)$ . Consider the image  $\pi(Z) \subset \mathbb{A}_S^1$  that is a closed subscheme since  $Z$  is finite over  $S$ , and moreover  $\pi(Z)$  is finite over  $S$ , and has one connected component. Hence  $\pi(Z)$  is local henselian scheme, since  $S$  is such. It follows

now that the induced morphism  $Z \rightarrow \pi(Z)$  is finite, etale morphism of local henselian schemes, and it is an isomorphism on closed points; thus  $\pi$  induces the isomorphism  $Z \rightarrow \pi(Z)$ .

Now we arguing like in [Mor1, Lemma 4.1.6]. Since  $\pi(Z) \subset \mathbb{A}_S^1$  is a closed subscheme finite over  $S$ , it is a closed subscheme in  $\mathbb{P}_S^1$  that does not meet the infinite section  $\infty_S \subset \mathbb{P}_S^1$ . Consider the sequence of equivalences in  $\mathbf{H}(S)$ :

$$(4) \quad U/(U - Z) \simeq \mathbb{A}_S^1/(\mathbb{A}_S^1 - \pi(Z)) \simeq \mathbb{P}_S^1/(\mathbb{P}_S^1 - \pi(Z))$$

Looking on the following diagram we see that the class of the canonical morphism  $U_+ \rightarrow \mathbb{P}_S^1/(\mathbb{P}_S^1 - Z)$  in  $\mathbf{H}_\bullet(S)$  is equal to the class of the morphism  $U \rightarrow S \xrightarrow{\infty} \mathbb{P}_S^1$

$$\begin{array}{ccccc} U & \xrightarrow{\quad} & \mathbb{A}_S^1 & \xrightarrow{\simeq} & \text{pt}_S \\ \downarrow & \searrow j & \nearrow \infty & \nearrow * & \nearrow \\ \mathbb{P}_S^1 & \xrightarrow{\quad} & \mathbb{P}_S^1/(\mathbb{P}_S^1 - \pi(Z)), & & \end{array}$$

where  $\infty: \text{pt}_S \rightarrow \mathbb{P}_S^1$  denotes the infinite section, and  $j$  is the immersion  $\mathbb{A}_S^1 \simeq \mathbb{P}_S^1 - \infty_S \rightarrow \mathbb{P}_S^1$ . Thus since  $\infty_S \subset \mathbb{P}_S^1 - \pi(Z)$  it follows that the class of the map  $U_+ \rightarrow \mathbb{P}_S^1/(\mathbb{P}_S^1 - \pi(Z))$  in the category  $\mathbf{H}_\bullet(S)$  is equal to the pointed one. Hence by the equivalences (4) the class of the canonical morphism  $U_+ \rightarrow U/(U - Z)$  is equal to the pointed morphism in the category  $\mathbf{H}_\bullet(S)$ . The claim follows.  $\square$

**Lemma 1.** *Let  $U$  be an essentially smooth local henselian scheme over a base scheme  $S$ . Then there is an etale map  $\pi: U \rightarrow \mathbb{A}_S^n$  that induces isomorphism  $x \rightarrow \pi(x)$ , where  $x \in U$  is the closed point, and  $n = \dim_S U$ .*

*Proof.* The base field case is well known and it is particular case of the Gabber's presentation lemma [Gab],[CTHK] and [HK].

Let  $\pi': U \times_S \sigma \rightarrow \mathbb{A}_\sigma^n$  be the required morphism over the closed point  $\sigma \in S$ . Then  $x \simeq \pi'(x)$ . Let  $\pi$  be any lift of  $\pi'$ . Then it follows that  $\pi$  is etale and  $x \simeq \pi(x)$ .  $\square$

**Lemma 2.** *Let  $S$  be a scheme. Let  $X$  be a smooth scheme of dimension  $n$  over a base scheme  $S$ , and let  $x \in X$  be a point over a point  $\sigma \in S$ . Let  $Z \subset X, Z \ni x$ , be a closed subscheme of a positive codimension over  $\sigma$ .*

*Then there regular map  $f = (f_1, \dots, f_{n-1}): X \rightarrow \mathbb{A}_S^{n-1}$  smooth at  $x$  and such that  $Z$  is of a positive codimension over  $p(x) \in \mathbb{A}_S^{n-1}$ .*

*Proof.* Without loss of generality we can assume that  $x \in X$  is a closed point (in a scheme of a finite type), and  $Z \subset X$  is a closed subscheme such that  $Z \times_S \sigma$  of a pure codimension one in  $X \times_S \sigma$ .

Since  $x$  is a smooth point it follows that there is a trivialisation  $\tau = (\tau_1, \dots, \tau_n): \mathcal{O}(x)^n \rightarrow \Omega_{x/X}$ . Denote by  $\Gamma$  the subset in  $\mathcal{O}(X)^{n-1}$  that consists of sets  $f = (f_1, \dots, f_{n-1})$  such that the differential of  $f$  is equal to  $\tau$ , i.e.  $df_i = \tau_i, i = 1, \dots, n-1$ . Then for any  $f \in \Gamma$  the morphism of schemes  $f: X \rightarrow \mathbb{A}_S^{n-1}$  is smooth at  $x$ .

We are going to construct by induction on  $i$  the functions  $f_1, \dots, f_i: X \rightarrow \mathbb{A}_S^i$  such that  $f_j|_x = 0, df_j = \tau_j, \forall j = 1 \dots i$ , and such that  $Z$  is of a positive codimension over the point  $0$  that is the image  $f^i(x) \in \mathbb{A}_S^i$  of  $x \in X$  under the morphism  $f^i = (f_1, \dots, f_i): X \rightarrow \mathbb{A}_S^i$ .

The base is the case of  $i = 0$ . Assume that we have constructed the functions  $f_1, \dots, f_i, i < n-1$ , we are going to construct  $f_{i+1}$ . Denote  $Z_i = Z(f_1, \dots, f_i) = Z \times_{\mathbb{A}_S^i} 0 \subset X \times_{\mathbb{A}_S^i} 0 \subset X$ . Then  $Z_i$  is of a pure codimension one in  $X \times_{\mathbb{A}_S^i} f^i(x)$  and of a pure codimension  $1+i$  in  $X$ . So  $\dim Z_i = n-1-i > 0$ . Choose a set of closed point  $S \subset Z_i$  such that  $x \ni S$  and  $S$  contains at least one point in each irreducible component of  $Z_i$ . Now choose  $f_{i+1}$  in a such way that  $f_{i+1}|_x = 0, f_{i+1}|_{Z(I^2(x))} = \tau_{i+1}$ , and  $f_i|_S = 1$ . Then it follows that  $f_i$  does not vanish on any irreducible component of  $Z_i$ , and hence  $Z(f_i) \cap Z_i$  is of a positive codimension in  $Z_i$ . Whence  $Z_{i+1} \stackrel{\text{def}}{=} Z(f_1, \dots, f_{i+1})$  is of a positive codimension in  $X \times_{\mathbb{A}_S^{i+1}} 0$ .

□

## 3. CONNECTIVITY THEOREM

**Theorem 3** (Proposition 3.1. [SS], Lemma 4.2.1 [Mor1]). *Let  $S$  be a base scheme. Let  $U \in \text{Sm}_S$  be a scheme of Krull dimension  $d$ . Let  $F \in \mathbf{SH}_{S^1}(S)$  be a connected spectrum. Then  $[U, F \wedge S^i]_{\mathbf{SH}_{S^1}(S)} = 0$  for all  $i > d$ .*

*Sketch of the proof.* Firstly let us recall the formula for the  $\mathbb{A}^1$ -localisation functor  $L_{\mathbb{A}^1}^{\text{nis}, S^1}$  on the category  $\mathbf{SH}_{\text{nis}}^{S^1}(S)$  of  $S^1$ -spectra of Nisnevich sheaves

$$L_{\mathbb{A}^1}^{\text{nis}, S^1}(F) = (\Omega_{S^1_{\mathbb{A}^1}}^\infty \Sigma_{S^1}^\infty F, \dots, \Omega_{S^1_{\mathbb{A}^1}}^\infty \Sigma_{S^1}^{\infty+i} F \dots), S_{\mathbb{A}^1}^1 = \Delta_{\mathbb{A}^1}^1 / \Delta_{\mathbb{A}^1}^0 = \mathbb{A}^1 / \{0, 1\},$$

and precisely the isomorphism

$$[U, E]_{\mathbf{SH}^{S^1}(S)} \stackrel{\text{def}}{=} [U, L_{\mathbb{A}^1}^{\text{nis}, S^1} E]_{\mathbf{SH}_{\text{nis}}^{S^1}(S)} \simeq \varinjlim_n [U \wedge (\mathbb{A}^1 / \{0, 1\})^n, E]_{\mathbf{SH}_{\text{nis}}^{S^1}(S)}.$$

Briefly speaking the formula and the isomorphism follows since  $\mathbb{S}_{\mathbb{A}^1}^1 = \mathbb{A}^1 / \{0, 1\}$  is the  $\mathbb{A}^1$ -homotopy equivalent model for the sphere  $S^1$ , and since the functor  $\Omega_{S^1_{\mathbb{A}^1}}^\infty$  (and  $\Sigma_{S^1}^\infty$ ) preserves the Nisnevich local objects in the category of  $S^1$ -spectra of Nisnevich sheaves.

By using of the Postnikov tower the case of a connective spectra of Nisnevich sheaves  $E$  is equivalent to the case of Eilenberg-MacLane spectrum  $E = \text{EM}(M)$  of an abelian Nisnevich sheaf  $M$ . Finally since we have the vanishing of the Nisnevich cohomologies

$$H_{\text{nis}}^i(X, M) = 0, \forall i > \dim X,$$

the claim follows.

We refer the reader to [Mor1, Section 4.3] and [SS, Section 3]. □

**Theorem 4.** *Let  $S$  be a scheme of a Krull dimension  $d$ . Let  $U$  be an essentially smooth local henselian scheme over a base scheme  $S$ . Let  $F \in \mathbf{SH}_{S^1}(S)$  be a connected spectrum. Then*

$$[U, F \wedge S^i]_{\mathbf{SH}_{S^1}(S)} = 0, [U, F \wedge S^i]_{\mathbf{SH}(S)} = 0 \forall i > d.$$

*Proof.* Without loss of generality we can assume that  $S$  is local henselian since  $U$  is such. Denote by  $E$  the set of generic points of  $U \times_S \sigma$ , where  $\sigma \in S$  is the closed point. Consider the semi-local essentially smooth scheme  $V \subset U$ :

$$V = \varprojlim_{Z \subset U, Z \cap E = \emptyset} (U - Z),$$

where the limit is over the set of all closed subschemes  $Z$  in  $U$  that does not meet  $E$ . In other words  $V$  is the complement to the all closed subschemes in  $U$  of a positive relative codimension over  $\sigma$ . Then since the codimension of  $\sigma$  in  $S$  is less or equal to  $d$ , the dimension of the scheme  $V$  is less of equal to  $d$ . Hence by theorem 3 it follows that

$$[V, F \wedge S^i]_{\mathbf{SH}_{S^1}(S)} = 0, \forall i > d, \text{ and } [V \wedge (\mathbb{P}^1 / \infty)^{\wedge l}, F \wedge S^i]_{\mathbf{SH}_{S^1}(S)} = 0, \forall i > d + l,$$

and hence

$$[V, F \wedge S^i]_{\mathbf{SH}_{S^1}(S)} = 0, [V, F \wedge S^i]_{\mathbf{SH}(S)} = 0, \forall i > d,$$

Finally, the claim follows since by theorem 2 the canonical maps

$$[U, F \wedge S^i]_{\mathbf{SH}_{S^1}(S)} \rightarrow [V, F \wedge S^i]_{\mathbf{SH}_{S^1}(S)}, [U, F \wedge S^i]_{\mathbf{SH}_{S^1}(S)} \rightarrow [V, F \wedge S^i]_{\mathbf{SH}(S)}$$

are injective. □

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