Некоторые задачи теории риска и их применение к моделям страхового регулирования

Исходным пунктом доклада являются исследования математических моделей планирования работы компаний, ведущих свой бизнес на конкурентном и регулируемом страховом рынке ([1], [2]). В них делается акцент на ценовой конкуренции, приводящей к миграции страхователей и к возможности возникновения страховых циклов, чреватых кризисами. При моделировании учитывается, что компании преследуют различные стратегические цели. Они меняются со временем, в зависимости от финансового положения компании и от состояния рынка. Связанные друг с другом, такие модели дают единую, интегральную модель долгосрочного управления компанией и ряд рекомендаций по регулированию работы как отдельных компаний, так и страхового рынка в целом.

Эти исследования привели к необходимости обратиться к вероятности разорения в традиционной модели теории риска. Для нее получено новое приближение в терминах обратного гауссовского распределения ([3]).

Вероятность разорения является характеристикой, выраженной в абсолютных, а не в денежных единицах. В практических приложениях удобнее использовать капитал неразорения, обеспечивающий для вероятности неразорения за конечное время заранее заданное значение. Это приводит к необходимости критического исследования используемых ныне мер риска (в том числе “Value-at-Risk”) и рассматривать задачу, обратную к задаче о пересечении границы, с этих позиций. Для капитала неразорения в традиционной модели теории риска получены новые приближения ([4]).

[1] *Малиновский В.К.* Модели долгосрочного страхового планирования. Ценовая конкуренция и регулирование финансовой устойчивости. — М. Янус-К, 2020.

[2] *Malinovskii, V.K.* Insurance Planning Models. Price Competition and Regulation of Financial Stability. World Scientific Publishers, Singapore, 2021.

[3] *Malinovskii, V.K.* Level–Crossing Problems and Inverse Gaussian Distributions. Closed–Form Results and Approximations. Chapman and Hall/CRC, Boca Raton, 2021.

[4] *Malinovskii, V.K.* Risk Measures and Insurance Solvency Benchmarks. Fixed–Probability Levels in Renewal Risk Models. Chapman and Hall/CRC, Boca Raton, 2021.

Часть 1: введение (кн. [1], [2])

$$\underbrace{\mathsf{w}^{[0]} \xrightarrow{\gamma^{[0]}} \mathsf{u}^{[0]} \Big(\overset{\mathcal{F}^{[1]}}{\rightsquigarrow} \Pi^{[1]} \boxed{\downarrow\uparrow} \mathcal{I}^{[1]} \Big) \xrightarrow{\pi^{[1]}} \mathsf{w}^{[1]}}_{\text{1-st year}} \cdots$$

$$\cdots \xrightarrow{\pi^{[k-1]}} \mathsf{w}^{[k-1]} \underbrace{\xrightarrow{\gamma^{[k-1]}} \mathsf{u}^{[k-1]} \Big(\overset{\mathcal{F}^{[k]}}{\rightsquigarrow} \Pi^{[k]} \boxed{\downarrow\uparrow} \mathcal{I}^{[k]} \Big) \xrightarrow{\pi^{[k]}} \mathsf{w}^{[k]}}_{k\text{-th year}} \cdots .$$

A typical dynamics of the underwriting cycle is the alternation of the two waves, one of which consists of a number of years of “hard”, or profitable market, and the other — of a number of years of “soft”, or unprofitable market. Themselves, these waves are divided into the following quarters:

Quarter $\mathcal{FH}$ (Falling hard market) The annual market prices $\Pi^{[k]}$, $k = 1, 2, \ldots$, are such that $\Pi^{[1]} \succcurlyeq \Pi^{[2]} \succcurlyeq \Pi^{[3]} \succcurlyeq \cdots \succcurlyeq \mathsf{E}Y$, where $\mathsf{E}Y$ is the marginal cost of insurance

Quarter $\mathcal{FS}$ (Falling soft market) The annual market prices $\Pi^{[k]}$, $k = 1, 2, \ldots$, are such that $\mathsf{E}Y \succcurlyeq \Pi^{[1]} \succcurlyeq \Pi^{[2]} \succcurlyeq \Pi^{[3]} \succcurlyeq \cdots$.

Quarter $\mathcal{RS}$ (Rising soft market) The annual market prices $\Pi^{[k]}$, $k = 1, 2, \ldots$, are such that $\Pi^{[1]} \preccurlyeq \Pi^{[2]} \preccurlyeq \Pi^{[3]} \preccurlyeq \cdots \preccurlyeq \mathsf{E}Y$.

Quarter $\mathcal{RH}$ (Rising hard market) The annual market prices $\Pi^{[k]}$, $k = 1, 2, \ldots$, are such that $\mathsf{E}Y \preccurlyeq \Pi^{[1]} \preccurlyeq \Pi^{[2]} \preccurlyeq \Pi^{[3]} \preccurlyeq \cdots$.

Assessment of financial position of a company in one year:

- Expansion
- Revenue
- Solvency

Часть 2 (кн. [3])

We denote the probability density function (p.d.f.) and cumulative distribution function (c.d.f.) of a standard Gaussian distribution by

$$\varphi_{(0,1)}(x) := \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}, \quad \Phi_{(0,1)}(x) := \int_{-\infty}^{x} \varphi_{(0,1)}(z)\, dz, \quad x \in \mathsf{R},$$

respectively.

Definition 1. *The compound renewal process with time $s \geqslant 0$ is*

$$V_s := \sum_{i=1}^{N_s} Y_i, \tag{1}$$

or 0, if $T_1 > s$, where summation is up to the random variable

$$N_s := \max\left\{n > 0 : \sum_{i=1}^{n} T_i \leqslant s\right\}, \tag{2}$$

with $N_s := 0$, if $T_1 > s$.

Plainly, the trajectories of the point process V_s, $s \geqslant 0$, and of its special case N_s, $s \geqslant 0$, called renewal process, are piecewise linear.

If all random variables T_i, $i = 1, 2, \ldots$, are identically distributed, the renewal process N_s, $s \geqslant 0$, is called ordinary. If $f_{T_1}(x)$ and $f_T(x)$ are not the same, this process is called modified renewal process. If $f_{T_1}(x)$ has the special form $f_{T_1}(x) := \frac{1}{\mathsf{E}T}(1 - F_T(x))$, where $F_T(x)$ is the cumulative distribution function (c.d.f.) corresponding to $f_T(x)$, then this process is called equilibrium (or stationary) renewal process.

By shifted compound renewal process, we refer to as $V_s - cs$, $s \geqslant 0$, where the underlying compound renewal process V_s, $s \geqslant 0$, is defined in (1) and $c > 0$ is a constant. We write

$$\varUpsilon_{u,c}^{[\mathrm{ren}]} := \inf\left\{s > 0 : V_s - cs > u\right\}, \tag{3}$$

or $+\infty$, if $V_s - cs \leqslant u$ for all $s > 0$. It is the first level-crossing time, or the first passage time to the level $u > 0$.

By the level-crossing problem in renewal framework, we mean an investigation of the r.v. $\varUpsilon_{u,c}^{[\mathrm{ren}]}$ defined in (3). This is the first time moment, when the shifted compound renewal process $V_s - cs$, $s \geqslant 0$, crosses the horizontal level u for the first time. Plainly, this problem can be reformulated in terms of the original compound renewal process V_s, $s \geqslant 0$. To do this, we need to make trivial changes in (3), i.e., write it as $\varUpsilon_{u,c}^{[\mathrm{ren}]} := \inf\{s > 0 : V_s > u + cs\}$, which means that we are focussed on a linearly growing, rather than horizontal, level.

Cramérian approximation. We denote by $M_X(r) := \mathsf{E}\, e^{rX}$ the moment generating function (m.g.f.) of X. Plainly, we have $M_X(0) = 1$. Let us consider the nonlinear equation (w.r.t. $r > 0$)

$$M_X(r) = 1 \tag{4}$$

and assume that a solution $\varkappa > 0$ to equation (4) exists[1]. This is a substantial restriction on the model which implies that $M_X(r)$ has to exist in a neighborhood of 0, and that the right tail of F_X is bounded above exponentially. The latter follows from Markov's inequality

$$1 - F_X(x) \leqslant e^{-\varkappa x}\, \mathsf{E}\, e^{\varkappa X} = e^{-\varkappa x}, \quad x > 0.$$

[1]In the risk theory, equation (4) is called Lundberg's equation. Its positive solution $\varkappa$ is called Lundberg's exponent, or adjustment coefficient. The assumption that it exists means that T, called interclaim time, may be either light-tailed, or heavy-tailed, whereas Y, called claim amount, must be light-tailed. This assumption is referred to as "small claim" condition.

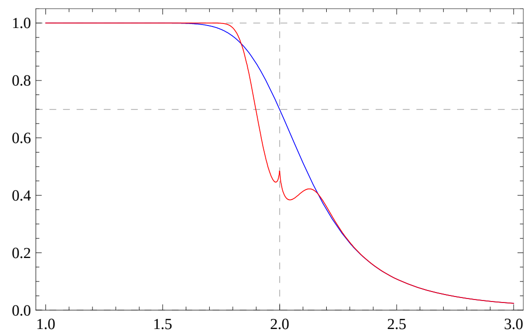

Рис. 1. Graph (X-axis is c) of $\mathsf{P}\{\varUpsilon^{[\mathrm{ren}]}_{u,c\,|\,\delta,\varrho} \leqslant t\}$ (solid line), which is a monotone decreasing function calculated numerically, using Type II formula (15), and the approximation of Theorem 2 (dash-dotted line), when T and Y are exponentially distributed with parameters $\delta = 2$ and $\varrho = 1$, respectively, and $t = 200$, $u = 10$. Horizontal line: $\mathsf{P}\{\varUpsilon^{[\mathrm{ren}]}_{u,c^*\,|\,\delta,\varrho} \leqslant t\} = 0.699$.

Starting with c.d.f. $F_{XT}(x,t) := \mathsf{P}\{X \leqslant x, T \leqslant t\}$, we proceed with the associated joint distribution, whose c.d.f. $F_{\bar{X}\bar{T}}(x,t) := \mathsf{P}\{\bar{X} \leqslant x, \bar{T} \leqslant t\}$ is defined by the equality[2]

$$F_{\bar{X}\bar{T}}(x,t) = \int_{-ct}^{x}\int_{0}^{t} e^{\varkappa z}\, F_{XT}(dz,dw). \tag{5}$$

Plainly, this is a proper probability distribution.

For $\bar{X}_i \stackrel{d}{=} \bar{X}$, $i = 1,2,\ldots$, and $\bar{T}_i \stackrel{d}{=} \bar{T}$, $i = 1,2,\ldots$, called associated random variables, we introduce $\bar{S}_n := \sum_{i=1}^{n} \bar{X}_i$, and $\bar{Z}_n := \sum_{i=1}^{n} \bar{T}_i$, $n = 1,2,\ldots$, referred to as associated random walks.

Theorem 1. *Assume that a solution $\varkappa > 0$ to equation (4) exists. For $\bar{\eta}(u) := \min\{n \geqslant 1 : \bar{S}_n > u\}$, or 0, if the random walk $\bar{S}_n$, $n = 1,2,\ldots$, which starts at the origin (i.e., with $\bar{S}_0 := 0$), never crosses the level $u > 0$, we have*

$$\mathsf{P}\{\varUpsilon^{[\mathrm{ren}]}_{u,c} \leqslant t\} = \mathsf{E}\, e^{-\varkappa \bar{S}_{\bar{\eta}(u)}}\, \mathbf{1}_{(-\infty,t]}(\bar{Z}_{\bar{\eta}(u)}). \tag{6}$$

$$\begin{aligned}
&\mathbb{C} = \delta/(c\varrho), && \varkappa = \varrho\,(1-\delta/(c\varrho)),\\
&m_{\triangledown} = -\frac{1}{c\,(1-\delta/(c\varrho))}, && D^2_{\triangledown} = -\frac{2\,(\delta/(c\varrho))}{c^2\varrho\,(1-\delta/(c\varrho))^3},\\
&m_{\vartriangle} = \frac{(\delta/(c\varrho)}{c\,(1-\delta/(c\varrho))}, && D^2_{\vartriangle} = \frac{2\,(\delta/(c\varrho))}{c^2\varrho\,(1-\delta/(c\varrho))^3}.
\end{aligned} \tag{7}$$

Theorem 2. *In the renewal model, when T and Y are exponentially distributed with parameters $\delta > 0$ and $\varrho > 0$, respectively, we have for $0 < c < c^* = \delta/\varrho$*

$$\sup_{t>0}\Big|\,\mathsf{P}\{\varUpsilon^{[\mathrm{ren}]}_{u,c\,|\,\delta,\varrho} \leqslant t\} - \varPhi_{(m_{\triangledown}u,D^2_{\triangledown}u)}(t)\,\Big| = o(1), \quad u \to \infty,$$

where $m_{\triangledown} > 0$, $D^2_{\triangledown} > 0$ are defined in (7), and for $c > c^*$

$$\sup_{t>0}\Big|\,e^{\varkappa u}\,\mathsf{P}\{\varUpsilon^{[\mathrm{ren}]}_{u,c\,|\,\delta,\varrho} \leqslant t\} - \mathbb{C}\,\varPhi_{(m_{\vartriangle}u,D^2_{\vartriangle}u)}(t)\,\Big| = o(1), \quad u \to \infty, \tag{8}$$

where $\varkappa > 0$, and $0 < \mathbb{C} < 1$, $m_{\vartriangle} > 0$, $D^2_{\vartriangle} > 0$ are defined in (7).

[2] A shorthand for this, $F_{\bar{X}\bar{T}}(dx,dt) = e^{\varkappa z} F_{XT}(dx,dt)$, is usually used.

Exact expression based on Kendall's identity. To separate T_1 apart from the rest[3], we introduce the conditional probability[4] $\mathsf{P}\{v < \varUpsilon^{[\mathrm{ren}]}_{u,c} \leqslant t \mid T_1 = v\}$, $0 < v < t$, and write

$$\begin{aligned}\mathsf{P}\{\varUpsilon^{[\mathrm{ren}]}_{u,c} \leqslant t\} = &\int_0^t \mathsf{P}\{u + cv - Y < 0\}\, f_{T_1}(v)\, dv \\ &+ \int_0^t \mathsf{P}\{v < \varUpsilon^{[\mathrm{ren}]}_{u,c} \leqslant t \mid T_1 = v\}\, f_{T_1}(v)\, dv.\end{aligned} \tag{9}$$

Let us introduce[5]

$$M_x := \inf\left\{k \geqslant 1 : \sum_{i=1}^{k} Y_i > x\right\} - 1, \quad x > 0, \tag{10}$$

which is a renewal process (cf. (2)) generated by the random variables Y_i, $i = 1, 2, \ldots$. The following identity is paramount.

THEOREM 3. *For $0 < v < t$, we have*

$$\begin{aligned}\mathsf{P}\{v < \varUpsilon^{[\mathrm{ren}]}_{u,c} \leqslant t \mid T_1 = v\} &= \int_v^t \frac{u + cv}{u + cz}\, \mathsf{p}_{\sum_{i=2}^{M_{u+cz}+1} T_i}(z - v)\, dz \\ &= \int_v^t \frac{u + cv}{u + cz} \sum_{n=1}^{\infty} \mathsf{P}\{M_{u+cz} = n\}\, f_T^{*n}(z - v)\, dz.\end{aligned} \tag{11}$$

Bearing in mind that Y_i, $i = 1, 2, \ldots$, are i.i.d., we have

$$\begin{aligned}\mathsf{P}\{M_{u+cv+cy} = n\} &= \mathsf{P}\left\{\sum_{i=1}^{n} Y_i \leqslant u + cv + cy < \sum_{i=1}^{n+1} Y_i\right\} \\ &= \int_0^{u+cv+cy} f_Y^{*n}(u + cv + cy - z)\, \mathsf{P}\{Y_{n+1} > z\}\, dz.\end{aligned}$$

Making the change of variables $y = z - v$ in (11), we rewrite it as

$$\begin{aligned}\mathsf{P}\{v < \varUpsilon^{[\mathrm{ren}]}_{u,c} \leqslant t \mid T_1 = v\} = &\sum_{n=1}^{\infty} \int_0^{t-v} \frac{u + cv}{u + cv + cy} \\ &\times \int_0^{u+cv+cy} \mathsf{P}\{Y_{n+1} > z\} \\ &\times f_Y^{*n}(u + cv + cy - z)\, f_T^{*n}(y)\, dy\, dz,\end{aligned} \tag{12}$$

where f_T^{*n} and f_Y^{*n} are n-fold convolutions of p.d.f. f_T and f_Y.

Exact formulas in Poisson–Exponential case.

THEOREM 4. *In the renewal model satisfying the standard assumptions, with T and Y exponentially distributed with parameters δ and ϱ, respectively, for $0 < v < t$ we have*

$$\begin{aligned}\mathsf{P}\{v < \varUpsilon^{[\mathrm{ren}]}_{u,c\,|\,\delta,\varrho} \leqslant t \mid T_1 = v\} = &\sqrt{\varrho\delta c}\,(v + u/c)\, e^{-\varrho u} e^{-c\varrho v} \\ &\times \int_0^{t-v} \frac{I_1(2\sqrt{\varrho\delta c(y + v + u/c)y}\,)}{\sqrt{(y + v + u/c)y}} \\ &\times e^{-(c\varrho+\delta)y}\, dy.\end{aligned} \tag{13}$$

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[3]When $T_1 \stackrel{d}{=} T$ is assumed, we restrict ourselves to the ordinary renewal process N_s, $s \geqslant 0$.

[4]Dealing with it, we are in the situation when at least one renewal occurs before time t.

[5]The inf-definition for M_x, $x > 0$, in contrast to max-definition (see (2)) for N_s, $s \geqslant 0$, is used here as a hint on the difference between these renewal processes.

THEOREM 5 (Type I formula). *In the renewal model satisfying the standard assumptions, with T and Y exponentially distributed with positive parameters δ and ϱ, we have*

$$\begin{aligned}\mathsf{P}\{\varUpsilon^{[\mathrm{ren}]}_{u,c\,|\,\delta,\varrho}\leqslant t\} = e^{-u\varrho}\delta\int_0^t e^{-(\varrho c+\delta)x}\Big(I_0(2\sqrt{\delta\varrho x\,(cx+u)}\,)\\ -\frac{cx}{cx+u}\,I_2(2\sqrt{\delta\varrho x\,(cx+u)}\,)\Big)\,dx.\end{aligned}\tag{14}$$

THEOREM 6 (Type II formula). *In the renewal model satisfying the standard assumptions, with T and Y exponentially distributed with positive parameters δ and ϱ, we have*

$$\begin{aligned}\mathsf{P}\{\varUpsilon^{[\mathrm{ren}]}_{u,c\,|\,\delta,\varrho}\leqslant t\} = e^{-u\varrho}\frac{\sqrt{\delta}}{\sqrt{c\varrho}}\int_0^{c\varrho t} e^{-(1+\delta/(c\varrho))x}\sum_{n=0}^{\infty}\frac{u^n}{n!}\left(\frac{\delta\varrho}{c}\right)^{n/2}\\ \times\frac{n+1}{x}\,I_{n+1}(2x\sqrt{\delta/(c\varrho)}\,)\,dx.\end{aligned}\tag{15}$$

THEOREM 7 (Type III formula). *In the renewal model satisfying the standard assumptions, with T and Y exponentially distributed with positive parameters δ and ϱ, we have*

$$\mathsf{P}\{\varUpsilon^{[\mathrm{ren}]}_{u,c\,|\,\delta,\varrho}\leqslant t\} = \mathsf{P}\{\varUpsilon^{[\mathrm{ren}]}_{u,c\,|\,\delta,\varrho}<\infty\} - \frac{1}{\pi}\int_0^{\pi} f(x,u,t)\,dx,\tag{16}$$

where

$$\mathsf{P}\{\varUpsilon^{[\mathrm{ren}]}_{u,c\,|\,\delta,\varrho}<\infty\} = \begin{cases}\dfrac{\delta}{c\varrho}\exp\{-u(c\varrho-\delta)/c\}, & \delta/(c\varrho)<1,\\ 1, & \delta/(c\varrho)\geqslant 1,\end{cases}$$

and

$$\begin{aligned}f(x,u,t) &= (\delta/(c\varrho))(1+\delta/(c\varrho)-2\sqrt{\delta/(c\varrho)}\cos x)^{-1}\\ &\quad\times\exp\Big\{u\varrho\,(\sqrt{\delta/(c\varrho)}\cos x-1)\\ &\quad - t\delta(c\varrho/\delta)(1+\delta/(c\varrho)-2\sqrt{\delta/(c\varrho)}\cos x)\Big\}\\ &\quad\times(\cos(u\varrho\sqrt{\delta/(c\varrho)}\sin x)-\cos(u\varrho\sqrt{\delta/(c\varrho)}\sin x+2x)).\end{aligned}$$

Inverse Gaussian Distribution. With the usual parametrization, p.d.f. of inverse Gaussian distribution is

$$\begin{aligned}f(x;\mu,\lambda,-\tfrac12) &:= \frac{\lambda^{1/2}}{\sqrt{2\pi}}\,x^{-3/2}\exp\Big\{-\frac{\lambda\,(x-\mu)^2}{2\,\mu^2 x}\Big\}\\ &= \lambda^{1/2}x^{-3/2}\,\varphi_{(0,1)}\left(\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu}-1\right)\right),\quad x>0.\end{aligned}\tag{17}$$

The parameters $\mu>0$ and $\lambda>0$ are called mean and shape parameters, respectively. The shape of p.d.f. $f(x;\mu,\lambda,-\frac12)$, as λ is fixed and μ grows, and as μ is fixed and λ grows, is illustrated in Fig. 2.

Obviously, the inverse Gaussian distribution concentrated on the positive half-line is not similar to the Gaussian distribution concentrated on the whole real line. But c.d.f. of the former,

$$F(x;\mu,\lambda,-\tfrac12) := \int_0^x f(z;\mu,\lambda,-\tfrac12)\,dz,\quad x>0,\tag{18}$$

can be expressed through c.d.f. of the latter. Let us express (18) first in terms of c.d.f. of a standard Gaussian distribution, and then in an equivalent form, i.e., in terms of Mills' ratio and p.d.f. of a standard Gaussian distribution.

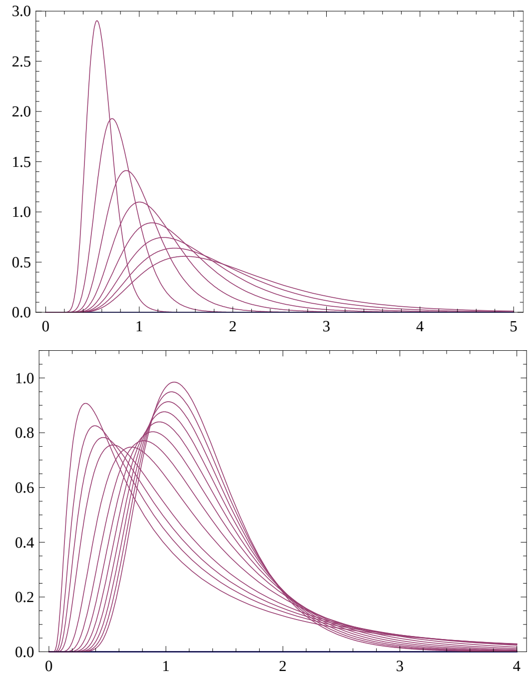

Рис. 2. Graphs (X-axis is x) of p.d.f. $f(x;\mu,\lambda,-\frac{1}{2})$ with (above) shape parameter $\lambda = 10$ and eight values of mean parameter: $\mu = 0.6$, 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, and 2.0 and (below) with mean parameter $\mu = 1.3$ and twelve values of shape parameter: $\lambda = 1.0$, 1.3, 1.6, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0, and 10.0.

Theorem 8. *The cumulative distribution function of inverse Gaussian distribution with parameters $\mu > 0$ and $\lambda > 0$ is*

$$\begin{aligned} F(x;\mu,\lambda,-\tfrac{1}{2}) = \varPhi_{(0,1)}\biggl(\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu}-1\right)\biggr) \\ + \exp\left\{\frac{2\lambda}{\mu}\right\}\varPhi_{(0,1)}\biggl(-\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu}+1\right)\biggr), \quad x > 0. \end{aligned} \tag{19}$$

Inverse Gaussian Approximation. The aim of this chapter, which is the nub of the book, is to get the inverse Gaussian approximation

$$\mathsf{P}\{\varUpsilon_{u,c}^{[\mathrm{ren}]} \leqslant t\} \approx \mathcal{M}_{u,c}(t), \quad u,t \to \infty, \tag{20}$$

where $\varUpsilon_{u,c}^{[\mathrm{ren}]}$ is defined in (3). This approximation is referred to as inverse Gaussian because of the approximating expression in (20), which is

$$\mathcal{M}_{u,c}(t) := \int_0^{\frac{ct}{u}} \frac{1}{x+1}\,\varphi_{\left(cM(x+1),\,\frac{c^2D^2}{u}(x+1)\right)}(x)\,dx,$$

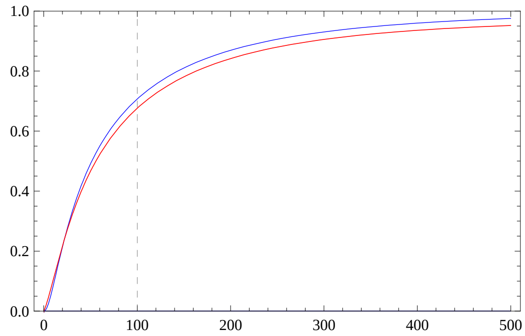

Рис. 3. Graphs (X-axis is t) of $\mathsf{P}\{v < \Upsilon^{[\mathrm{ren}]}_{u,c\,|\,\delta,\varrho} \leqslant t \mid T_1 = v\}$ (solid line), calculated numerically, using formula (13), and $\mathcal{M}_{u,c}(t \mid v)$ (dash-dotted line), when T and Y are exponentially distributed with parameters $\delta = \varrho = 1$ (whence $c^* = 1$), and $v = 0$, $u = 10$, $c = 0.9$.

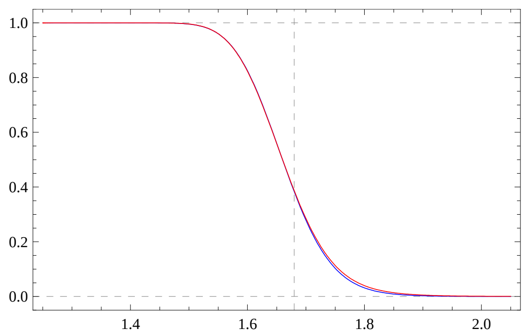

Рис. 4. Graphs (X-axis is c) of $\mathsf{P}\{\Upsilon^{[\mathrm{ren}]}_{u,c\,|\,\delta,\varrho} \leqslant t\}$ (solid line), calculated numerically, using formula (14), and $\mathcal{M}_{u,c}(t)$ (dash-dotted line), when T and Y are exponentially distributed with parameters $\delta = 2.1$ and $\varrho = 1.25$, respectively, and $t = 800$, $u = 40$. Vertical grid line is $c^* = \delta/\varrho = 1.68$.

where $M := \mathsf{E}T/\mathsf{E}Y$, $D^2 := ((\mathsf{E}T)^2\mathsf{D}Y + (\mathsf{E}Y)^2\mathsf{D}T)/(\mathsf{E}Y)^3$. It may be written in terms of c.d.f. of inverse Gaussian distribution, as follows:

$$\mathcal{M}_{u,c}(t) = \begin{cases} \Big(F(x+1;\mu,\lambda,-\tfrac{1}{2}) \\ \quad -F(1;\mu,\lambda,-\tfrac{1}{2})\Big)\Big|_{x=\frac{ct}{u},\mu=\frac{1}{1-cM},\lambda=\frac{u}{c^2D^2}}, & 0 < c < c^* := M^{-1}, \\ \exp\Big\{-\dfrac{2\lambda}{\hat{\mu}}\Big\}\Big(F(x+1;\hat{\mu},\lambda,-\tfrac{1}{2}) \\ \quad -F(1;\hat{\mu},\lambda,-\tfrac{1}{2})\Big)\Big|_{x=\frac{ct}{u},\hat{\mu}=\frac{1}{cM-1},\lambda=\frac{u}{c^2D^2}}, & c > c^*. \end{cases}$$

For $c > 0$, $u > 0$, $0 < v < t$, $c^* := \mathsf{E}Y/\mathsf{E}T$, and for

$$M := \mathsf{E}T/\mathsf{E}Y, \quad D^2 := ((\mathsf{E}T)^2\mathsf{D}Y + (\mathsf{E}Y)^2\mathsf{D}T)/(\mathsf{E}Y)^3, \tag{21}$$

we write

$$\mathcal{M}_{u,c}(t \mid v) := \int_0^{\frac{c(t-v)}{u+cv}} \frac{1}{x+1}\, \varphi_{\left(cM(x+1),\,\frac{c^2D^2}{u+cv}(x+1)\right)}(x)\, dx.$$

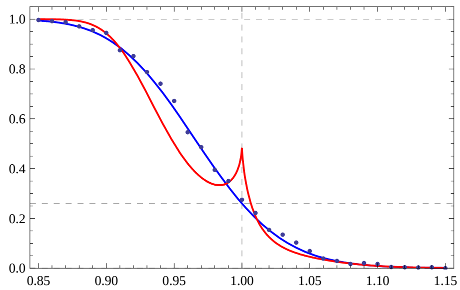

Рис. 5. Graphs (X-axis is c) of numerically evaluated $\mathsf{P}\{\varUpsilon_{u,c}^{[\mathrm{ren}]} \leqslant t\}$ (smooth solid line), of the approximations of Proposition 2 (non-smooth solid line), and simulated values ($\Delta c = 0.05$, $N = 1000$) of $\mathsf{P}\{\varUpsilon_{u,c}^{[\mathrm{ren}]} \leqslant t\}$ (dots), when T and Y are exponentially distributed with parameters $\varrho = 1$, $\delta = 1$, respectively, and $t = 1000$, $u = 50$. Horizontal grid line: $\mathsf{P}\{\varUpsilon_{u,c^*}^{[\mathrm{ren}]} \leqslant t\} = 0.26$.

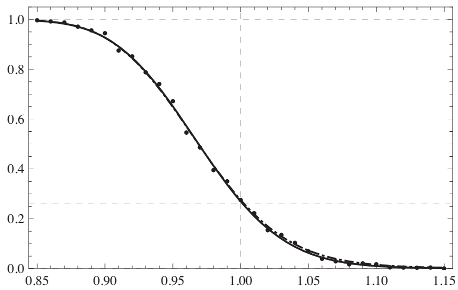

Рис. 6. Graphs (X-axis is c) of numerically evaluated $\mathsf{P}\{\varUpsilon_{u,c}^{[\mathrm{ren}]} \leqslant t\}$ (solid line), of $\mathcal{M}_{u,c}(t)$ (dash-dotted line), and simulated values ($\Delta c = 0.05$, $N = 1000$) of $\mathsf{P}\{\varUpsilon_{u,c}^{[\mathrm{ren}]} \leqslant t\}$, when T and Y are exponentially distributed with parameters $\delta = 1$, $\varrho = 1$, respectively, and $t = 1000$, $u = 50$. Horizontal grid line: $\mathsf{P}\{\varUpsilon_{u,c^*}^{[\mathrm{ren}]} \leqslant t\} = 0.26$.

Theorem 9. *In the renewal model, let p.d.f. f_T and f_Y be bounded above by a finite constant, $D^2 > 0$, $\mathsf{E}(T^3) < \infty$, $\mathsf{E}(Y^3) < \infty$. Then for any fixed $c > 0$ and $0 < v < t$, we have*

$$\sup_{t>v} \big| \mathsf{P}\{v < \varUpsilon_{u,c}^{[\mathrm{ren}]} \leqslant t \mid T_1 = v\} - \mathcal{M}_{u,c}(t \mid v) \big| = O\left(\frac{\ln(u+cv)}{u+cv}\right), \tag{22}$$

as[6] $u + cv \to \infty$.

Часть 3 (кн. [4])

In the models (a)–(c), the risk reserve process, which is a difference of incoming premiums and outgoing claims, is defined as

$$R_s = u + cs - V_s, \quad s \geqslant 0, \tag{23}$$

where $u \geqslant 0$ is the initial capital, or the initial risk reserve, $c \geqslant 0$ is the premium intensity, called for brevity price,

$$V_s = \sum_{i=1}^{N_s} Y_i, \tag{24}$$

[6] With c and v fixed, $u + cv \to \infty$ is trivially equivalent to $u \to \infty$.

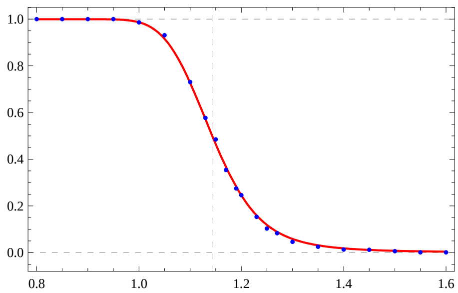

Рис. 7. Graph (X-axis is c) of numerically evaluated $\mathcal{M}_{u,c}(t)$ (solid line) and simulated values ($\Delta c = 0.05$, $N = 1000$) of $\mathsf{P}\{\, \varUpsilon^{[\text{ren}]}_{u,c} \leqslant t\}$ (dots), when T is 2-mixture and Y is Pareto (see details in Table 1) $t = 1000$, $u = 40$.

or 0, if $N_s = 0$ (or $T_1 > s$), is the aggregate claim payout process, and

$$N_s = \max\left\{ n > 0 : \sum_{i=1}^{n} T_i \leqslant s \right\}, \tag{25}$$

or 0, if $T_1 > s$, is the claim arrival process.

Definition 2 (Non-loss capital). *The non-loss capital $\underline{u}_{\alpha,t}(c)$, $c \geqslant 0$, is a positive solution to the equation (w.r.t. u)*

$$\mathsf{P}\{V_t - ct \leqslant u\} = 1 - \alpha. \tag{26}$$

We set $\underline{u}_{\alpha,t}(c)$ equal to zero for those c, for which this solution is negative.

Definition 3 (Non-ruin capital). *The non-ruin capital $u_{\alpha,t}(c)$, $c \geqslant 0$, is a positive solution to the equation (w.r.t. u)*

$$\mathsf{P}\Big\{ \sup_{0 \leqslant s \leqslant t} (V_s - cs) \leqslant u \Big\} = 1 - \alpha. \tag{27}$$

We set $u_{\alpha,t}(c)$ equal to zero for those c, for which this solution is negative.

In risk theory, common is to write (27) as

$$\mathsf{P}\Big\{ \inf_{0 \leqslant s \leqslant t} R_s < 0 \Big\} = \alpha, \tag{28}$$

in terms of the risk reserve process (23). Plainly, the left-hand side of (28) is the probability of ruin within time t. Using (28), the origin of the term "non-ruin capital" is straightforward: if $u_{\alpha,t}(c)$ is chosen to be the initial capital in the expression (23) for the risk reserve, then the probability of ruin $\psi_t(u, c)$ is equal to α.

Bounds on non-ruin capital, when $0 \leqslant c \leqslant c^*$. We start with the following bilateral bounds, when $0 \leqslant c \leqslant c^*$:

$$\begin{aligned}(c^* - c)\, t + \frac{D}{M^{3/2}}\, \kappa_\alpha \sqrt{t}\, (1 + o(1)) &\leqslant u_{\alpha,t}(c) \\ &\leqslant (c^* - c)\, t + \frac{D}{M^{3/2}}\, \kappa_{\alpha/2} \sqrt{t}\, (1 + o(1)), \quad t \to \infty.\end{aligned} \tag{29}$$

Bounds on non-ruin capital, when $c > c^*$. We proceed with the upper bounds, when $c > c^*$. In this case, sensible is to start with $\overline{u}_\alpha(c)$, $c > c^*$, which yields an accurate upper bound, when c is much larger than c^*, and a very inaccurate upper bound, when c^* is a little less than c. We further correct this bound, balancing precision and complexity, and get a number of upper bounds for all $c > c^*$. Finally, we compare these bounds with the simulated values of $u_{\alpha,t}(c)$, $c > c^*$.

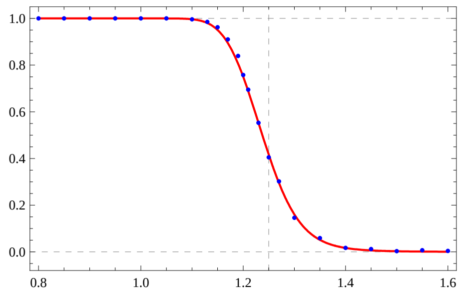

Рис. 8. Graph (X-axis is c) of numerically evaluated $\mathcal{M}_{u,c}(t)$ (solid line) and simulated values ($\Delta c = 0.05$, $N = 1000$) of $\mathsf{P}\{\varUpsilon^{[\mathrm{ren}]}_{u,c} \leqslant t\}$ (dots), when T is Erlang and Y is Pareto (see details in Table 1), $t = 1000$, $u = 40$.

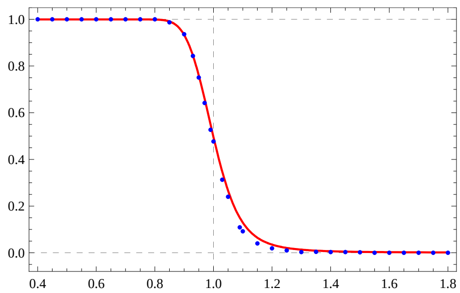

Рис. 9. Graph (X-axis is c) of numerically evaluated $\mathcal{M}_{u,c}(t)$ (solid line) and simulated values ($\Delta c = 0.05$, $N = 1000$) of $\mathsf{P}\{\varUpsilon^{[\mathrm{ren}]}_{u,c} \leqslant t\}$ (dots), when T and Y are Pareto (see details in Table 1), $t = 1000$, $u = 40$.

Таблица 1. Models in Figs. 6–9

	inter-claim interval T	claim amount Y	M	D^2
Fig. 6:	exponentially distributed; $\delta = 1$	exponentially distributed; $\varrho = 1$	1	2
Fig. 7:	2-mixture; $\delta_1 = 1$, $\delta_2 = 2$, $p = 2/3$	Pareto; $a_Y = 4.0$, $b_Y = 0.35$	0.88	2.30
Fig. 8:	Erlang; $\delta = 6.0$, $k = 4$	Pareto; $a_Y = 4.0$, $b_Y = 0.4$	0.8	1.2
Fig. 9:	Pareto; $a_T = 4.0$, $b_T = 0.4$	Pareto; $a_Y = 4.0$, $b_Y = 0.4$	1	1.33

Seeking for more or less elementary, but accurate, upper bounds for $u_{\alpha,t}(c)$, $c > c^*$, we focus first on its natural upper bound $\overline{u}_\alpha(c)$, $c > c^*$. The latter is a solution to equation

$$\mathsf{P}\{\varUpsilon^{[\mathrm{ren}]}_{u,c} < \infty\} = \alpha,$$

where $\varUpsilon^{[\mathrm{ren}]}_{u,c} := \inf\{s \geqslant 0 : V_s - cs > u\}$.

The left-hand side of this equation is the well-studied probability of ultimate ruin[7]. Following the theory developed for $\mathsf{P}\{\varUpsilon^{[\mathrm{ren}]}_{u,c} < \infty\}$, when $c > c^*$, we focus on the following particular models.

[7]We recall that $\boldsymbol{\psi}_\infty(u,c) := \mathsf{P}\{\varUpsilon^{[\mathrm{ren}]}_{u,c} < \infty\}$, or $\boldsymbol{\psi}_\infty(u,c) := \mathsf{P}\{\inf_{s\geqslant 0} R_s < 0\}$.

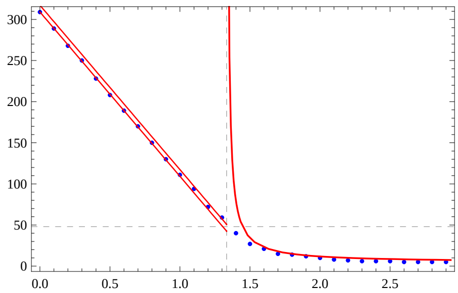

Рис. 10. Graphs (X-axis is c) of two-sided bounds (29), when $0 \leqslant c \leqslant c^*$, upper bound, when $c > c^*$, and simulated values of $u_{\alpha,t}(c)$ (dots) in Model (D), *Example (b)*, i.e., for T Erlang with parameters $\delta = 8/5$, $k = 2$, and Y exponentially distributed with parameter $\varrho = 3/5$, $\alpha = 0.05$, $t = 200$. Vertical grid line: $c^* = 4/3$. Horizontal grid line: $u_{\alpha,t}(c^*) = 48$.

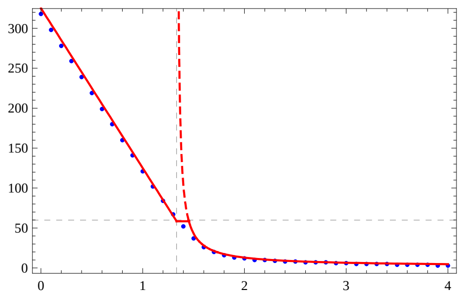

Рис. 11. Graphs (X-axis is c) of upper bound (29), when $0 \leqslant c \leqslant c^*$, upper bound (31), when $c > c^*$, and simulated values of $u_{\alpha,t}(c)$ (dots) in Model (A), i.e., for T and Y exponentially distributed with parameters $\delta = 3/5$, $\varrho = 4/5$, respectively, and $\alpha = 0.05$, $t = 200$. Vertical grid line: $c^* = 4/3$. Horizontal grid line: $u_{\alpha,t}(c^*) = 59.90$.

Model (A): compound Poisson model, when Y is exponentially distributed,

Model (B): compound Poisson model, when distribution of Y is light-tailed (but not exponential),

Model (C): compound Poisson model, when distribution of Y is fat-tailed,

Model (D): renewal (but not compound Poisson) model, when Y is exponentially distributed,

Model (E): renewal (but not compound Poisson) model, when distribution of Y is light-tailed (but not exponential),

Model (F): renewal (but not compound Poisson) model, when the distribution of Y is fat-tailed.

Model (A). This model is verbally described as a model with generic inter-claim interval T and claim size Y both exponentially distributed, with positive parameters δ and ϱ, respectively. The assumption on T can be also formulated as that claims arrival process is Poisson, with intensity δ.

By elementary calculations, we have $c^* = \delta/\varrho$, $M = \varrho/\delta$, $D^2 = 2\,\varrho/\delta^2$. The adjustment coefficient, or Lundberg's exponent, is $\varkappa = \varrho - \delta/c$.

When $c > \delta/\varrho$, we have

$$\mathsf{P}\{\varUpsilon_{u,c}^{[\mathrm{ren}]} < \infty\} = (1 - \varkappa/\varrho)\, e^{-\varkappa\, u}, \quad c > \delta/\varrho,$$

for all $u \geqslant 0$. This can be rewritten as[8]

$$\mathsf{P}\{\varUpsilon_{u,c}^{[\mathrm{ren}]} < \infty\} = (\delta/(c\varrho))\, \exp\{-(\varrho - \delta/c)\, u\}, \quad c > \delta/\varrho, \tag{30}$$

for all $u \geqslant 0$, and by elementary calculations we have

$$u_{\alpha,t}(c) \leqslant \max\left\{0, -\frac{\ln(\alpha c\varrho/\delta)}{\varrho - \delta/c}\right\}, \quad c > \delta/\varrho, \tag{31}$$

which is an upper bound that satisfies our needs.

In Fig. 11, the upper bounds (29) in the case $0 \leqslant c \leqslant \delta/\varrho$, and (31) in the case $c > \delta/\varrho$, are drawn for $t = 200$, $\alpha = 0.05$, $\delta = 4/5$, $\varrho = 3/5$, whence $c^* = 1.3333$, $M = 0.75$, and $D^2 = 1.875$. By dots are shown the simulated values of $u_{\alpha,t}(c)$.

Note that in Fig. 11 a bound slightly to the right of the point $c^* = \delta/\varrho$ is shown as a horizontal line at the level $u_{\alpha,t}(c^*) = 59.9033$, up to its intersection with $\overline{u}_\alpha(c)$, $c > \delta/\varrho$. This is a justified upper bound on $u_{\alpha,t}(c)$, which is obviously monotone decreasing, as c increases. If we prove (which is uneasy) that $u_{\alpha,t}(c)$ is convex[9], then we would construct more accurate bounds.

Let us make a remark concerning (see Proposition 2) the normal approximation, which turns out to be useless for the above analysis. In Model (A), when $c > c^* = \delta/\varrho$, we have

$$\lim_{u\to\infty} \mathsf{P}\{\varUpsilon_{u,c}^{[\mathrm{ren}]} < \infty\}\, e^{\varkappa\, u} = \mathbb{C}, \tag{32}$$

where $0 < \mathbb{C} = \delta/(c\varrho) < 1$ and $\varkappa = \varrho - \delta/c > 0$.

Nobody can deny that (32) is one of the most famous and best known results of risk theory. This result holds not only in Model (A), but in much more general renewal risk models. But the information on the asymptotic behavior of $\mathsf{P}\{\varUpsilon_{u,c}^{[\mathrm{ren}]} < \infty\}$, as $u \to \infty$, is pointless in the study of $u_{\alpha,t}(c)$, when $c > Kc^*$, $K > 1$, as above, because in this case $u_{\alpha,t}(c)$ is finite.

Model (B). This model is verbally described as a model with generic inter-claim interval T exponentially distributed with parameter δ, and claim size Y whose distribution is light-tail, but non-exponentially distributed, e.g.,

Example (a): T is exponentially distributed and Y is 2-mixture,

Example (b): T is exponentially distributed and Y is Erlang.

The assumption on T can be also formulated as that claims arrival process is Poisson, with intensity δ.

Since $c^* = \delta\, \mathsf{E}\, Y$, equation (4) can be rewritten as the equation (w.r.t. r) $\mathsf{E} \exp\{r\, Y\} = 1 + c\, r/\delta$, whose positive solution $\varkappa$ is the adjustment coefficient, or Lundberg's exponent. When $c > c^*$, we have

$$\mathsf{P}\{\varUpsilon_{u,c}^{[\mathrm{ren}]} < \infty\} \leqslant e^{-\varkappa\, u}, \quad c > \delta\, \mathsf{E}\, Y, \tag{33}$$

for all $u \geqslant 0$. Therefore, by simple calculations we have

$$u_{\alpha,t}(c) \leqslant -\frac{\ln \alpha}{\varkappa}, \quad c > \delta\, \mathsf{E}\, Y, \tag{34}$$

and the problem reduces to finding $\varkappa$ in a closed form.

Let us make a remark concerning the balance between efficiency and simplicity. In Model (A), to get the upper bound (31) for $u_{\alpha,t}(c)$, $c > \delta/\varrho$, we used a closed-form expression (30) for $\mathsf{P}\{\varUpsilon_{u,c}^{[\mathrm{ren}]} < \infty\}$. In Model (B), which includes Model (A) as a special

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[8] Recall that $\overline{u}_\alpha(c)$ is a solution to equation $\mathsf{P}\{\varUpsilon_{u,c}^{[\mathrm{ren}]} < \infty\} = \alpha$. In this case, this equation can be rewritten as $(\delta/(c\varrho))\, \exp\{-(\varrho - \delta/c)\, u\} = \alpha$.

[9] Convex functions are of the shape $\smile$.

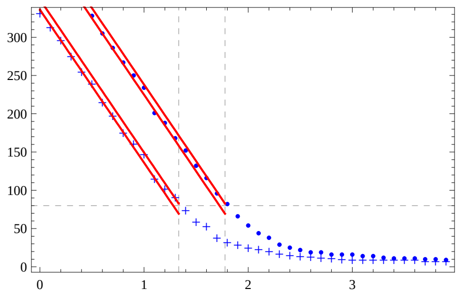

Рис. 12. Graphs (X-axis is c) of two-sided bounds (29), when $0 \leqslant c \leqslant c^*$, and simulated values of $u_{\alpha,t}(c)$ in Model (C), *Example (a)*, i.e., for T exponentially distributed with parameter $\delta = 4/5$ and Y Pareto with parameters $a_Y = 10$, $b_Y = 0.05$ (dots), $a_Y = 3$, $b_Y = 0.3$ (crosses), and $\alpha = 0.05$, $t = 200$. Vertical grid lines: $c^* = 1.78$ (dots) and $c^* = 1.33$ (crosses). Horizontal grid line: $u_{\alpha,t}(c^*) = 80$ (the same for dots and crosses).

case, we used inequality (33): in this case there are no such accurate and simple-looking results as (30). Consequently, the upper bound (34), which for Y exponentially distributed with parameter ϱ takes the form

$$u_{\alpha,t}(c) \leqslant -\frac{\ln \alpha}{\varrho - \delta/c}, \quad c > \delta/\varrho, \tag{35}$$

is less accurate than (31). Thus, sacrificing accuracy for generality, we strive not to sacrifice simplicity.

Model (C). This model is verbally described as a model with generic inter-claim interval T exponentially distributed with parameter δ, and claim size Y whose distribution is fat-tail, e.g.,

Example (a): T is exponentially distributed and Y is Pareto,

Example (b): T is exponentially distributed and Y is Kummer.

The assumption on T can be also formulated as that claims arrival process is Poisson, with intensity δ.

In this case, we have no inequality like (33), but we have its substitutes of a rather complex structure.

To set an example, let us focus on Model (C), *Example (a)*: T is exponentially distributed with parameter $\delta > 0$ and Y is Pareto with parameters $a_Y > 0$, $b_Y > 0$, whose p.d.f. are

$$f_T(x) = \delta\, e^{-\delta\, x}, \quad f_Y(x) = \frac{a_Y b_Y}{(x\, b_Y + 1)^{a_Y + 1}}, \quad x > 0,$$

respectively. We can show by elementary calculations that

$$\begin{aligned} &\mathsf{E}\, T = 1/\delta, && \mathsf{D}\, T = 1/\delta^2, \\ &\mathsf{E}\, Y = 1/((a_Y - 1)\, b_Y), && \mathsf{D}\, Y = a_Y/((a_Y - 1)^2\, (a_Y - 2)\, b_Y^2). \end{aligned}$$

Plainly, $c^* = \delta/((a_Y - 1)\, b_Y)$,

$$M = \frac{(a_Y - 1)\, b_Y}{\delta}, \quad D^2 = \frac{2\,(a_Y - 1)^2\, b_Y}{\delta^2\,(a_Y - 2)},$$

and the adjustment coefficient does not exist.

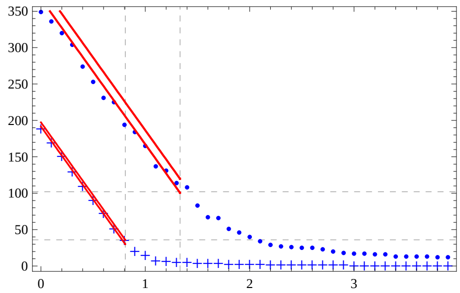

Рис. 13. Graphs (X-axis is c) of two-sided bounds (29), when $0 \leqslant c \leqslant c^*$, and simulated values of $u_{\alpha,t}(c)$ in Model (C), *Example (b)*, i.e., for T exponentially distributed with parameter $\delta = 4/5$ and Y Kummer with parameters $k_Y = 5$, $l_Y = 5$ (dots), $k_Y = 200$, $l_Y = 200$ (crosses), and $\alpha = 0.05$, $t = 200$. Vertical grid line: $c^* = 1.33$ (dots) and $c^* = 0.81$ (crosses). Horizontal grid lines: simulated $u_{\alpha,t}(c^*) = 102$ (dots) and $u_{\alpha,t}(c^*) = 36$ (crosses).

In Fig. 12, the upper and lower bounds (29) in the case $0 \leqslant c \leqslant c^*$ are drawn. By dots, drawn are simulated values of $u_{\alpha,t}(c)$, $c \geqslant 0$. We note that for $a_Y = 3$, $b_Y = 0.3$ (crosses), the third moment $\mathsf{E}(Y^3)$ is infinite, and the moment conditions may be relaxed.

To set another example, let us focus on Model (C), *Example (b)*: T is exponentially distributed with parameter $\delta > 0$ and Y is Kummer with parameters $k_Y > 0$, $l_Y > 0$, whose p.d.f. are

$$f_T(x) = \delta\, e^{-\delta\, x}, \quad f_Y(x) = \frac{k_Y}{2} \frac{\Gamma\big(\frac{k_Y + l_Y}{2}\big)}{\Gamma\big(\frac{k_Y}{2}\big)}\, U\bigg(1 + \frac{l_Y}{2}, 2 - \frac{k_Y}{2}, \frac{k_Y}{l_Y}\, x\bigg), \quad x > 0,$$

with $U(a,b,z) = \Gamma(a)^{-1} \int_0^\infty e^{-zt} t^{a-1} (1+t)^{b-a-1}\, dt$, respectively. We can show by elementary calculations that

$$\mathsf{E}(T^k) = \frac{k!}{\delta^k}, \quad \mathsf{E}(Y^k) = \frac{\Gamma\big(\frac{k_Y}{2} + k\big)\Gamma\big(\frac{l_Y}{2} - k\big)}{\Gamma\big(\frac{k_Y}{2}\big)\Gamma\big(\frac{l_Y}{2}\big)}\, l_Y^k\, k_Y^{-k}, \quad 2k < l_Y, \quad k = 1, 2, \ldots.$$

In particular,

$$\mathsf{E}\, T = 1/\delta, \qquad \mathsf{D}\, T = 1/\delta^2,$$
$$\mathsf{E}\, Y = \frac{l_Y}{l_Y - 2}, \quad \mathsf{D}\, Y = \frac{l_Y^2 (\, 4(\, l_Y - 2) + k_Y l_Y)}{k_Y (\, l_Y - 2)^2 (\, l_Y - 4)}.$$

Plainly, $c^* = \delta\, l_Y / (l_Y - 2)$,

$$M = \frac{l_Y - 2}{\delta\, l_Y}, \quad D^2 = \frac{2\,(2 + k_Y)(\, l_Y - 2)^2}{\delta^2 k_Y (l_Y - 4)\, l_Y},$$

and the adjustment coefficient does not exist.

In Fig. 13, the upper and lower bounds (29) in the case $0 \leqslant c \leqslant c^*$ are drawn. Bounds, when $c > c^*$, are beyond the scope of this article and are not considered, although the essence of the complexity in their construction is clear. By dots, drawn are the simulated values of $u_{\alpha,t}(c)$, $c \geqslant 0$.

Let us make a final remark concerning analytical results in the case $c > c^*$. Although the analytical aspects of the problem of finding $\overline{u}_\alpha(c)$ in Model (C) are conceptually clear, the analytical solution is extremely difficult to present in an observable form, as it was done in Model (A). This entails a computational awkwardness, rather than complexity, which is easy to overcome by a sacrifice of elegance of the bound, e.g., its smoothness and even strict monotone decrease, as c increases[10].

Two approaches to numerical calculations are as follows: if an analytical expression for $\mathsf{P}\{\varUpsilon_{u,c}^{[\mathrm{ren}]} < \infty\}$ is available, then $u_{\alpha,t}(c)$ is evaluated by numerically solving the equation $\mathsf{P}\{\varUpsilon_{u,c}^{[\mathrm{ren}]} < \infty\} = \alpha$ at every point c of interest. Otherwise, $u_{\alpha,t}(c)$ is obtained by Monte Carlo simulation.

Model (D). This model is verbally described as a model with renewal claims arrival, i.e., the distribution of T is non-exponential, and with claim size Y exponentially distributed with parameter ϱ, e.g.,

Example (a): T is 2-mixture and Y is exponentially distributed,

Example (b): T is Erlang and Y is exponentially distributed,

Example (c): T is Pareto and Y is exponentially distributed,

Example (d): T is Kummer and Y is exponentially distributed.

Plainly, we have $c^* = 1/(\varrho\,\mathsf{E}\,T)$, equation (4) can be rewritten as the equation (w.r.t. r) $\mathsf{E}\exp\{-r\,c\,T\} = 1 - r/\varrho$, and $\varkappa$ is its positive solution. When $c > c^*$, we have $\mathsf{P}\{\varUpsilon_{u,c}^{[\mathrm{ren}]} < \infty\} = (1 - \varkappa/\varrho)\,e^{-\varkappa u}$ for all $u \geqslant 0$. Bearing in mind that $1 - \varkappa/\varrho \leqslant 1$, we have

$$u_{\alpha,t}(c) \leqslant -\ln\alpha/\varkappa, \quad c > 1/(\varrho\,\mathsf{E}\,T), \tag{36}$$

and the problem reduces to finding $\varkappa$ in a closed form.

To set an example, let us focus on[11] Model (D), *Example (b)*: T is Erlang with parameters k integer and $\delta > 0$ and Y is exponentially distributed with parameter $\varrho > 0$.

We can show by elementary calculations that $c^* = \delta/(k\varrho)$,

$$M = k\varrho/\delta, \quad D^2 = k\,(k+1)\,\varrho/\delta^2,$$

and, when $c > \delta/(k\varrho)$, the positive solution $\varkappa$ to Lundberg's equation (4), which can be rewritten as the equation (w.r.t. r)

$$(\varrho - r)\,(\delta + c\,r)^k - \delta^k\,r = 0,$$

is easy to find numerically.

In Fig. 10, the upper and lower bounds (29) in the case $0 \leqslant c \leqslant \delta/\varrho$, and the upper bound (36) in the case $c > \delta/\varrho$, are drawn for $t = 200$, $\alpha = 0.05$, $\delta = 8/5$, $k = 2$, $\varrho = 3/5$, whence $c^* = 1.3333$, $M = 0.75$, and $D^2 = 1.40625$. In Fig. 10, by dots are drawn the simulated values of $u_{\alpha,t}(c)$.

Model (E). This model is verbally described as a model with renewal claims arrival, i.e., the distribution of T is non-exponential, and with claim size Y whose distribution is light-tail, but not exponential, e.g.,

Example (a): T is Erlang and Y is Erlang,

Example (b): T is Erlang and Y is 2-mixture.

We address $X \overset{d}{=} Y - c\,T$, whose c.d.f. is F_X, denote by $\overline{F}_X(x) = 1 - F_X(x)$ is tail function, and write $x_0 = \sup\{x : F_X(x) < 1\}$. When $c > c^* := \mathsf{E}\,Y/\mathsf{E}\,T$, we have

$$b_{\ominus}\,e^{-\varkappa u} \leqslant \mathsf{P}\{\varUpsilon_{u,c}^{[\mathrm{ren}]} < \infty\} \leqslant b_{\oplus}\,e^{-\varkappa u} \quad \text{for all} \quad u \geqslant 0, \tag{37}$$

where $\varkappa$ is a positive solution to (4) and

$$b_{\oplus} = \inf_{x\in[0,x_0]} \frac{e^{\varkappa x}\overline{F}_X(x)}{\int_x^\infty e^{\varkappa y}\,dF_X(y)}, \quad b_{\ominus} = \sup_{x\in[0,x_0]} \frac{e^{\varkappa x}\overline{F}_X(x)}{\int_x^\infty e^{\varkappa y}\,dF_X(y)}.$$

[10] Simply speaking, the bound may be horizontal in some places.

[11] Note that this model is a particular case of Model (E), where T is Erlang with parameters k integer and $\delta > 0$ and Y is Erlang with parameters m integer and $\varrho > 0$

Alternatively, the inequalities (37) hold true with

$$b_{\oplus}^{*} = \inf_{x\in[0,x_0^*]} \frac{e^{\varkappa x}\overline{F}_Y(x)}{\int_x^\infty e^{\varkappa y}\, dF_Y(y)}, \quad b_{\ominus}^{*} = \sup_{x\in[0,x_0^*]} \frac{e^{\varkappa x}\overline{F}_Y(x)}{\int_x^\infty e^{\varkappa y}\, dF_Y(y)},$$

where $x_0^* = \sup\{x : F_Y(x) < 1\}$; the inequalities $0 \leqslant b_{\ominus}^{*} \leqslant b_{\ominus} \leqslant b_{\oplus} \leqslant b_{\oplus}^{*} \leqslant 1$ hold.

Both upper and lower bounds for $\overline{u}_\alpha(c)$, $c > c^*$, which is a solution to equation $\mathsf{P}\{\varUpsilon_{u,c}^{[\mathrm{ren}]} < \infty\} = \alpha$, and therefore upper bounds for $u_{\alpha,t}(c)$, $c > c^*$, is easy to get from (37), at least numerically.

Model (F). This model is verbally described as a model with renewal claims arrival, i.e., the distribution of T is non-exponential, and with claim size Y, whose distribution is fat-tail, e.g.,

Example (a): T is 2-mixture and Y is Pareto,

Example (b): T is Erlang and Y is Pareto,

Example (c): T is Pareto and Y is Pareto.

Regarding this model, we repeat that any upper bound for $u_{\alpha,t}(c)$, $c > c^*$, which assumes *small, rather than large* values, is tightly related to the probability $\mathsf{P}\{\varUpsilon_{u,c}^{[\mathrm{ren}]} < \infty\}$ for *small, rather than large* values of u.