Вспомним то, что было.

$(\mathfrak{X}, \Sigma, \mu)$; $L^1(\mu)$ — пр-во суммируемых функций.

норма, полнота, аппроксимация.

$\|f\| = \int |f| d\mu.$

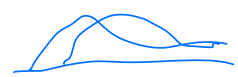
$\mathfrak{X} = \mathbb{R}$, $\mu = \lambda$:

сдвиги, $f(t) \mapsto f(t-a)$

непрерывность.

$\|f(t) - f(t-a)\| \to 0$

$a \to \infty$



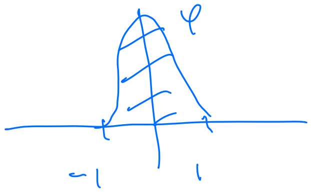
Свертка: $f, g \to (f*g)(t) = \int\limits_{\mathbb{R}} f(t-\tau)\, g(\tau)\, d\tau$

$\|f*g\| \le \|f\| \cdot \|g\|.$

Упражнения: $f*g = g*f$;

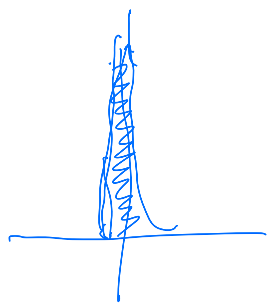
$f*(g_1+g_2) = f*g_1 + f*g_2.$

Аппроксимативная единица.



$\int \varphi(t)\,dt = 1$

$\operatorname{supp}\varphi \subset (-1,1) \qquad \varphi(t) \in [0,1]$.



$\varepsilon > 0$

$\frac{1}{\varepsilon}\,\varphi\left(\frac{t}{\varepsilon}\right) = \varphi_\varepsilon(t)$

$f \in L^1(\mathbb{R})$

$f_\varepsilon(t) = (\varphi_\varepsilon * f)(t) \qquad \{\varphi_\varepsilon\}_{\varepsilon>0}$

Утверждение: $\|f_\varepsilon - f\| \to 0$, $\varepsilon \to 0$.

Д-во: $f_\varepsilon(t) = \int_{-\infty}^{\infty} f(t-\tau)\,\varphi_\varepsilon(\tau)\,d\tau$

$-$ $f(t) = \int_{-\infty}^{\infty} f(t)\,\varphi_\varepsilon(\tau)\,d\tau$.

$f_\varepsilon(t) - f(t) = \int_{-R} \left[f(t-\tau) - f(t)\right] \varphi_\varepsilon(\tau)\,d\lambda \qquad |\tau| < \varepsilon$.

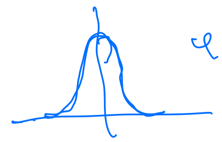
$\forall \delta \ \exists \varepsilon:$

ε-дост. малым: $\|f(t-\tau) - f(t)\| < \delta$

$\tau < \varepsilon$.

$$\|f_\varepsilon(t) - f(t)\| \le \delta \underbrace{\int \varphi_\varepsilon(\tau)\,d\tau}_{=1}.$$

Преимущество:



$\varphi \in C_0^\infty(\mathbb{R})$

$$f_\varepsilon(t) = \int f(\tau)\,\varphi_\varepsilon(t-\tau)\,d\tau.$$

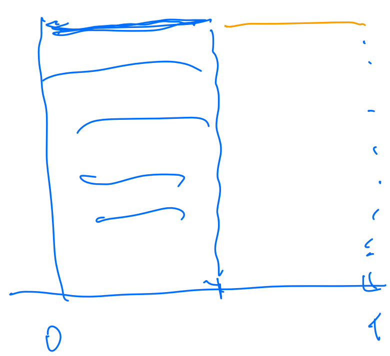
$$\Delta \sum f(\tau_k)\,\varphi_\varepsilon(t-\tau_k)$$

$f \in L^1([0,1])$, $\quad f_n \xrightarrow{L^1} f.$

Вопрос: $f_n(t) \to f(t)$ почти всюду?

Не обязательно!

Пример:

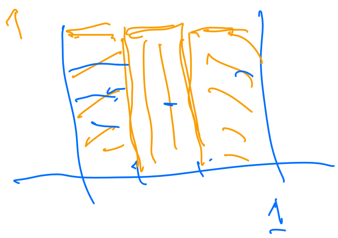


$f(t) = 0$

$\|f_n\| \to 0$

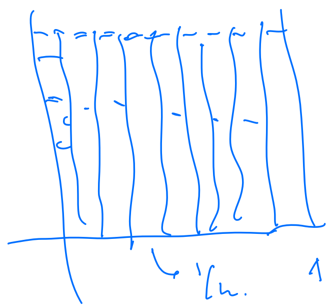
но $f_n \not\to 0$ п.в.

$n = 2$



$\frac{1}{3}$

$n = 3$



$\frac{1}{n}$

n^2

Как $f_n \to f$ если $\| f - f_n \| \to 0$?

Нер-во Чебышева. $\Rightarrow$

$$\Rightarrow \lambda(\{x : |f_n(x) - f(x)| < \varepsilon\}) \le$$

$$\le \frac{1}{\varepsilon} \|f_n - f\| .$$

<u>Опр:</u> $f_n \xrightarrow{\mu} f$, $n \to \infty$

Если $\forall \varepsilon > 0$, $\mu(\{x : |f - f_n| \geq \varepsilon\}) \to 0$, $n \to \infty$

<u>Утв. 1</u> $f_n \to f$ – почти всюду

Тогда $f_n \xrightarrow{\mu} f$.

<u>Док-во:</u> Надо: $\forall \varepsilon > 0$
$$\mu(\{x : |f - f_n| > \varepsilon\}) \to 0 \quad ?$$
$$n \to \infty$$

$$R^n_\varepsilon = \{x : |f - f_n| > \varepsilon\} \leftarrow$$

$$T^\varepsilon_k = \bigcup_{n \geq k} R^\varepsilon_n, \qquad T^\varepsilon_k \supset T^\varepsilon_{k+1}$$

$$x \in \bigcap T^\varepsilon_k \to f_n(x) \not\to f(x)$$

$$x \in T^\varepsilon_k \Rightarrow \exists\, n > k : |f_n(x) - f(x)| > \varepsilon.$$

$$\Rightarrow \mu\left(\bigcap T^\varepsilon_k\right) = 0 \Rightarrow$$

$$\left.\begin{aligned} &\Rightarrow \mu(T^\varepsilon_k) \to 0, \\ &R^\varepsilon_k \subset T^\varepsilon_k \end{aligned}\right\} \quad k \to \infty.$$

$$\Longrightarrow \mu(R^\varepsilon_k) \to 0, \quad k \to \infty \quad \cancel{?}.$$

$(\mathfrak{X}, \Sigma, \mu)$

~~На всякий случай: $\mu(\mathfrak{X}) < \infty$.~~

<u>Утверждение:</u> $f_n \xrightarrow{\mu} f \Rightarrow \exists \{n_k\}$: $f_{n_k} \to f$ почти всюду.

<u>Д-во:</u> $f_n \xrightarrow{\mu} f$

$\Rightarrow \forall k \quad \exists n_k \qquad n_{k+1} > n_k$.

$$\mu\left(\left\{x : |f_i - f| > \frac{1}{k}\right\}\right) \le \frac{1}{2^{k+1}}$$

$i \ge n_k$.

<u>Докажем:</u> $\{f_{n_k}\}$ — то что надо!

$f_{n_k} \to f$, $k \to \infty$ п. в.

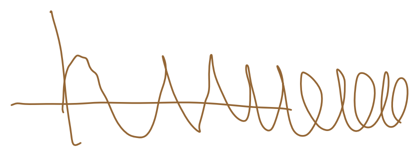
$\left\{x : |f_i - f| > \frac{1}{k}\right\}$

$h_k = |f_{n_k} - f| \qquad \tilde{h}_k = \sup\limits_{s \ge k} h_s$

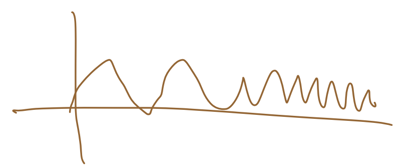
$h_k(x) \to 0$; $\quad \tilde{h}_k \downarrow$.

<u>Отступление:</u>

$f(x) = \sin x^2$



$\int_0^N \sin x^2 dx \to \int_0^\infty dx$



$$\mu(\{\tilde{h}_k > \varepsilon\}) \to 0, \quad k \to \infty$$

$$E_k = \{x : |f_i - f| > \tfrac{1}{k}\}, \quad i \geq n_k$$

$$\mu(E_k) \leq \frac{1}{2^k}.$$

$$\mu\Big(\bigcup_{k \geq n} E_k\Big) \leq \frac{1}{2^{n-1}}$$

$x : \overline{\lim_k} |f_{n_k}(x) - f(x)| > \varepsilon; \quad l \quad \frac{1}{l} < \varepsilon.$

$$x \in \underbrace{\bigcup_{k \geq l} E_k}_{R_k} \text{ при всех } l : \tfrac{1}{l} < \varepsilon$$

$$x \in \bigcap_{n \geq \frac{1}{\varepsilon}} R_k, \quad \mu(R_k) < \frac{1}{2^k}.$$

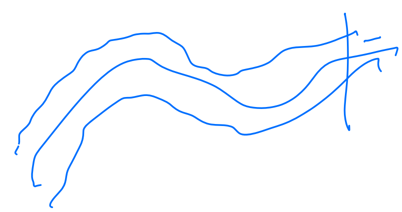
$$\mu\Big(\bigcap R_k\Big) = 0.$$

Теорема Егорова

$\mu(\mathfrak{X}) < \infty \qquad f_n \to f$ — почти всюду.

$\forall \delta > 0 \quad \exists E \subset \mathfrak{X}, \quad \mu(\mathfrak{X} \setminus E) < \delta$

и $\quad f_n \big|_E \rightrightarrows f \big|_E$



Док-во:

$$E_n^m = \{x : |f_i - f| < \tfrac{1}{m};\ i \geq n\}$$

$$E_n^m \subset E_{n+1}^m; \qquad E^m = \bigcup_n E_n^m.$$

$x \notin E^m \Rightarrow \exists$ сколь угодно большой i:

$$|f_i(x) - f(x)| > \tfrac{1}{m}.$$

$$\Rightarrow \mu(E^m) = \mu(\mathfrak{X})$$

$$\mu(E_n^m) \nearrow \mu(E^m) = \mu(\mathfrak{X})$$

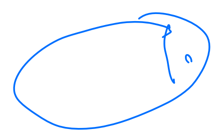
$\delta > 0$ Для m выберем $n(m)$,

$$\mu(\underbrace{E^m \setminus E^m_{n(m)}}_{A_m}) < \frac{\delta}{2^{m+1}}$$

$$\mu\left(\bigcup A_m\right) < \delta$$

$$x \notin \bigcup A_m \Rightarrow$$

$$\Rightarrow x \in \bigcap_m E^m_{n(m)}$$



$$x \in E^m_{n(m)} \qquad |f_{n(m)}(x) - f(x)| < \frac{1}{m}$$

$$\mu\left(\bigcap_m E^m_{n(m)}\right) = \mu(\mathfrak{X}) - \mu\left(\bigcup_m A_m\right) \geq$$

$$\geq \mu(\mathfrak{X}) - \delta.$$

Упр: Теорема Лузина.

След. эквивалентны:

1. f — измерима на $[0,1]$

2. $\forall \delta \ \exists E \subset [0,1]$, и $g \in C[0,1]$, такие что:

- $\mu(E) > 1 - \delta$
- $f\big|_E = g\big|_E$.

L^p $(\mathfrak{X}, \Sigma, \mu)$ — μ - σ-конечна

$\mathfrak{X} = \bigcup \mathfrak{X}_n$, $\mu(\mathfrak{X}_n) < \infty$.

$$L^p(\mathfrak{X}, \Sigma, \mu) = \left\{ f : \int_{\mathfrak{X}} |f(x)|^p \, d\mu < \infty \right\}.$$

$$\|f\|_p := \left(\int |f(x)|^p \, d\mu \right)^{1/p}$$

Вспомогательные нер-ва:

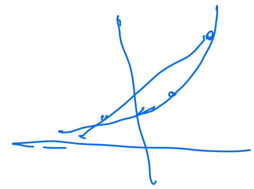
1. **Юнг** $\quad p, q \geq 1; \quad \frac{1}{p} + \frac{1}{q} = 1$

$$q = \frac{p-1}{p}.$$

Нер-во:

$x, y > 0 \quad |xy| \leq \frac{x^p}{p} + \frac{y^q}{q}.$

$f(t) = e^t$



$\alpha, \beta > 0 \qquad \alpha + \beta = 1$

$u, v \geq 0$

$e^{\alpha u + \beta v} \leq \alpha e^u + \beta e^v$

$$\boxed{x, y > 0 \qquad |xy| \leq \frac{x^p}{p} + \frac{y^q}{q}.}$$

$\alpha = \frac{1}{p}, \qquad \beta = \frac{1}{q}.$

$e^u = x^p \qquad u = p \log x$

$v = q \log y.$

Показатель

$\alpha u + \beta v = \log xy.$

Нер-во Гельдера:

$f, g, \quad \frac{1}{p} + \frac{1}{q} = 1$

$$\int_{\mathcal{X}} |f(x) g(x)| d\mu \leq \left(\int_{\mathcal{X}} |f(x)|^p d\mu \right)^{1/p} \cdot \left(\int_{\mathcal{X}} |g(x)|^q d\mu \right)^{1/q}.$$

Будем считать, что

$$\int |f(x)|^p d\mu = 1$$

$$\int |g(x)|^q d\mu = 1$$

Нам надо:

$$\int_{\mathcal{X}} |f(x)\, g(x)|\, d\mu \le 1$$

Юнг:

$$|f(x)\, g(x)| \le \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q$$

$$\int_{\mathcal{X}} |f(x) g(x)|\, d\mu \le$$

$$\le \frac{1}{p}\underbrace{\int_{\mathcal{X}} |f(x)|^p d\mu}_{=1} + \frac{1}{q}\underbrace{\int_{\mathcal{X}} |g(x)|^q d\mu}_{=1}$$

$$= 1.$$

Нер-во Минковского

$p > 1$, $f, g \in L^p$.

Тогда:

$$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

Док-во:

- $f, g \in L^p \Rightarrow f+g \in L^p$

докажите сами.

$$|f+g| \le 2\max\{|f|, |g|\}.$$

Дост. док. $\underbrace{\int \max\{|f|, |g|\}^p d\mu}$ $< \infty$.

$$\underbrace{\quad} = \int_{\{|f|\ge|g|\}} |f|^p d\mu + \int_{|g|>|f|} |g|^p d\mu \le$$

$$\le \int_{\mathcal{X}} |f|^p d\mu + \int_{\mathcal{X}} |g|^p d\mu.$$

- Собственно нер-во:

$p > 1$

$$\int_{\mathfrak{X}} |f+g|^p \, d\mu \leq$$

$$\leq \int_{\mathfrak{X}} \underbrace{|f|}_{p} \underbrace{|f+g|^{p-1}}_{q} \, d\mu + \underline{\int_{\mathfrak{X}} |g| \, |f+g|^{p-1} \, d\mu}$$

$$\int_{\mathfrak{X}} |f| \, |f+g|^{p-1} \, d\mu \leq \left(\int_{\mathfrak{X}} |f|^p \, d\mu \right)^{1/p} \cdot$$

$$\cdot \left(\int_{\mathfrak{X}} \big(\underbrace{|f+g|^{p-1}} \big)^q \, d\mu \right)^{1/q}.$$

$$\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow q = \frac{p}{p-1}$$

$$\left(|f+g|^{p-1} \right)^q = |f+g|^p$$

$$\underline{\int_{\mathfrak{X}} |f| \, |f+g|^{p-1} \, d\mu} \leq \left(\int |f|^p \, d\mu \right)^{\frac{1}{p}} \cdot$$

$$\left(\int |f+g|^p \, d\mu \right)^{\frac{p-1}{p}}$$

$$\underline{\int_{\mathfrak{X}} |g| \, |f+g|^{p-1} \, d\mu} \leq$$

$$\leq \left(\int |g|^p \, d\mu \right)^{\frac{1}{p}} \underline{\left(\int |f+g|^p \, d\mu \right)^{\frac{p-1}{p}}}.$$

1

$$\int |f+g|^p d\mu \le$$

$$\le \left[\left(\int |f|^p d\mu\right)^{\frac{1}{p}} + \left(\int |g|^p d\mu\right)^{\frac{1}{p}}\right] \underline{\left(\int |f+g|^p d\mu\right)^{\frac{p-1}{p}}}$$

$$1 - \frac{p-1}{p} = 1 - 1 + 1/p = 1/p.$$

$$\left(\int |f+g|^p d\mu\right)^{1/p} \le$$

$$\le \left(\int |f|^p d\mu\right)^{\frac{1}{p}} + \left(\int |g|^p d\mu\right)^{\frac{1}{p}}$$

Как связаны пр-ва L^p при разных p

$p' < p''$ $\quad L^{p'} \subset L^{p''}$

<u>Утв.:</u>

$$\left.\begin{array}{l} \mu(X) < \infty \\ p' < p'' \end{array}\right\} \Rightarrow L^{p''} \subset L^{p'}$$

В самом деле.

$$\int_{\mathfrak{X}} |f|^{\alpha_1}\, d\mu = \int_{\mathfrak{X}} |f|^{\alpha_1} \cdot 1 \, d\mu$$

$p = \dfrac{\alpha_2}{\alpha_1}$, $\quad \alpha_2 > \alpha_1$

$\alpha_2 > \alpha_1$

$p > 1$

$$\int_{\mathfrak{X}} |f|^{\alpha_1} \cdot 1 \, d\mu \le$$

$$\le \left(\int_{\mathfrak{X}} |f|^{\alpha_1 \cdot \frac{\alpha_2}{\alpha_1}} d\mu \right)^{1/p} \cdot \left(\int_{\mathfrak{X}} 1 \, d\mu \right)^{1/q} =$$

$$= \left(\int_{\mathfrak{X}} |f|^{\alpha_2} \right)^{\alpha_1/\alpha_2} \cdot \mu(\mathfrak{X})^{q_-}$$

$\alpha_1 \le \alpha_2$; $\quad \mu(\mathfrak{X}) < \infty$

$$\|f\|_{\alpha_2} < \infty \Rightarrow \|f\|_{\alpha_1} < \infty.$$

$$L^{\alpha_2} \subseteq L^{\alpha_1}$$

Упр: Это не так если $\mu(\mathfrak{X}) = \infty$.