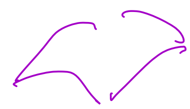
Интегрирование по поверхностям

Задача: $\mathbb{R}^m \supset U \xrightarrow{\varphi} \mathbb{R}^n$, $n > m$

$\varphi(U) = S$



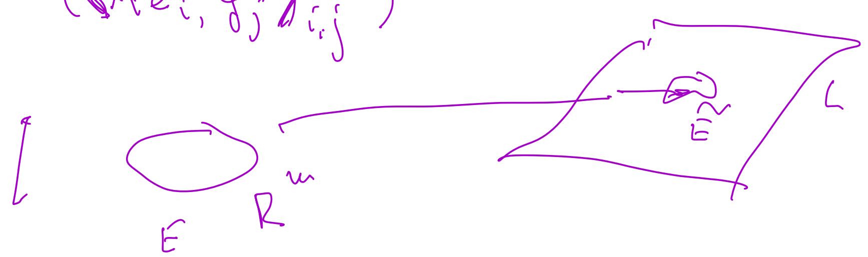
Как определить λ на S.

Случай 1 $A : \mathbb{R}^m \to \mathbb{R}^n$ линейное. $A\mathbb{R}^m =: \underline{L}$

$e_1, \dots, e_m$ — орт. базис в $\mathbb{R}^m$

$g_1 \dots g_m$ —"— —"— L

$\left(\langle Ae_i, g_j \rangle_{i,j} \right) = T$



$$\lambda_m(\tilde{E}) = |\det T| \, \lambda_m(E).$$

Задача: Независимость от $\{g_j\}$.

$$|\det(T^T T)|^{1/2} = \det T$$

$\tau_{ik} = (Ae_i, g_k)$ $\qquad T$

$\omega_{jk} = \tau_{kj} = (Ae_k, g_j)$

$$(T^T T)_{m,k} = \sum_i \underbrace{\langle Ae_m, g_i\rangle \langle Ae_k, g_i\rangle}$$

$$= \langle Ae_m, Ae_k\rangle$$

$$\Gamma(A) = (\langle Ae_m, Ae_k\rangle_{m,k}).$$

$$\det T = (\det \Gamma(A))^{1/2}.$$

Упр.: 1) $\Gamma(A) \geq 0$

2) $\det(\Gamma(A))$ не зависит от $\{e_i\}$.

Общий случай

$$\varphi : U \to R^n, \quad U \subset R^m$$

$\varphi = (\varphi_1, \dots, \varphi_n)$ $\qquad x_1, \dots, x_m$

$$D_\varphi(x) = \begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1} & \dots & \frac{\partial \varphi_1}{\partial x_m} \\ \frac{\partial \varphi_n}{\partial x_1} & & \frac{\partial \varphi_n}{\partial x_m} \end{pmatrix}(x)$$

$$\varphi : R^m \to R^n$$

$$\varphi(x+\Delta) = D_\varphi(x)\,\Delta + o(\Delta).$$

$$n\begin{pmatrix} D_\varphi \end{pmatrix}^{m}\begin{pmatrix} \; \end{pmatrix}^{m} \qquad \begin{pmatrix} \; \end{pmatrix}^{n}$$

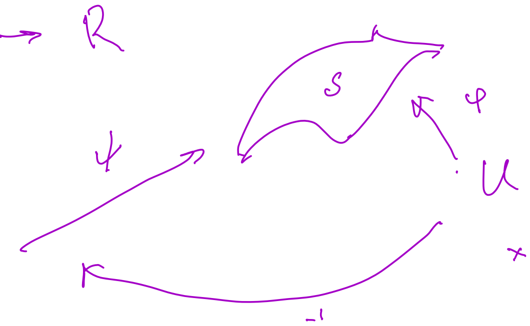
$$E \subset U \subset \mathbb{R}^m$$

$$\lambda_m(\varphi(E)) = \int_E \det \Gamma(D_\varphi(x))^{1/2}\, dx$$

$$\varphi(E) = \tilde{E} \subset S$$

$$\boxed{\underline{\lambda_m}(\tilde{E}) := \int_{\varphi^{-1}(\tilde{E})} \left(\det \Gamma(D_\varphi(x))\right)^{1/2} dx.}$$

$$\varphi : U \to \mathbb{R}^n$$
$$U \subset \mathbb{R}^m$$



$$y \qquad U_1 \subset \mathbb{R}^m$$

$$\beta = \psi^{-1}\varphi$$

$$dy = \det|\Delta_\beta|\, dx.$$

$$\lambda_m(\tilde{E}) = \int_{\psi^{-1}(E)} \det \underline{\Gamma(D_y(\psi))}^{1/2}\, dy.$$

$$D_\varphi(x) = \begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1} & \cdots & \frac{\partial \varphi_1}{\partial x_m} \\ \frac{\partial \varphi_n}{\partial x_1} & & \frac{\partial \varphi_n}{\partial x_m} \end{pmatrix}(x)$$

$$D_\psi(y) =$$

$$\Delta_x(\beta) = \begin{pmatrix} \frac{\partial \beta_1}{\partial x_1} & \frac{\partial \beta_1}{\partial x_m} \\ \frac{\partial \beta_m}{\partial x_1} & \frac{\partial \beta_m}{\partial x_m} \end{pmatrix}$$

$$D_x(\varphi) = D_y(\psi)\, D_x(\beta), \quad y = \beta(x)$$

$$\Gamma(D_x(\varphi)) = \left(D_x(\varphi)\, D_x(\varphi)^T\right) =$$

$$= \left(D_y(\psi)\, \underbrace{\Delta_x(\beta)\, \Delta_x(\beta)^T}\, D_y(\psi)\right)$$

$$\boxed{\det \Gamma(D_x(\varphi))^{1/2} = |\det \Delta_x| \left(\det \Gamma(D_y(\psi))\right)^{1/2}.}$$

$$\lambda_m(\tilde{E}) = \int_{\psi^{-1}(E)} \det \Gamma(D_y(\psi))^{1/2}\, dy. \qquad ?$$

$$\Uparrow ?$$

$$\boxed{\lambda_m(\tilde{E}) := \int_{\varphi^{-1}(\tilde{E})} \left(\det \Gamma(D_\varphi(x))\right)^{1/2} dx.}$$

$$dy = \det D_f \cdot dx.$$

$$\varphi : U \longrightarrow \mathbb{R}^n$$
$$U \subset \mathbb{R}^m$$

$$S = \varphi(U).$$

1) $\varphi : U \to S$ – взаимно однозначно

2) φ – гладкая.

3) $\forall x \quad D_\varphi(x) = \begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_m} \\ \frac{\partial \varphi_m}{\partial x_1} & \frac{\partial \varphi_n}{\partial x_m} \end{pmatrix}$

$\operatorname{rank} D_\varphi(x) = m.$

$$\varphi^{-1} : S \to U,$$

$$\psi : U_1 \to S$$

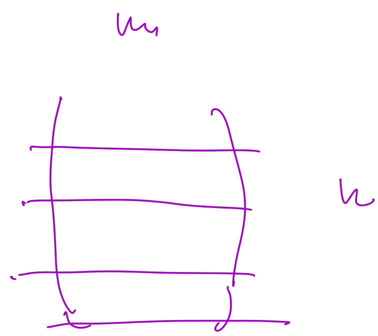
$$U_1 \xrightarrow{\psi} S \xrightarrow{\varphi^{-1}} U$$

$$\varphi^{-1}\psi = \gamma$$

$$\operatorname{ранг}\begin{pmatrix} \frac{\partial \varphi^1}{\partial y_1} & \dots & \frac{\partial \varphi^1}{\partial y_m} \\ \dots & & \dots \\ \frac{\partial \varphi^m}{\partial y_1} & \dots & \frac{\partial \varphi^m}{\partial y_m} \end{pmatrix} = m.$$

$$\psi^{-1}\varphi.$$

Бине-Коши:



$i_1, \dots, i_m$

$\delta_{i_1 \dots i_m} =$

$= \det(\text{строки } i_1, \dots, i_m \text{ в } B).$

Формула:

$$\det BB^T = \sum_{i_1 < \dots < i_m} \delta_i^2$$

$\boxed{m = n-1}$ $\quad \varphi: U \to \mathbb{R}^n$, $\quad U \subset \mathbb{R}^{n-1}$

$S = \varphi(U)$

$$D_\varphi(x) = \begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_{n-1}} \\ \vdots & \\ \frac{\partial \varphi_n}{\partial x_1} & \frac{\partial \varphi_n}{\partial x_{n-1}} \end{pmatrix} \; n$$

$$\begin{array}{c} n-1 \\ \Delta_i = \det \begin{pmatrix} \quad \\ \cancel{\qquad\qquad} \\ \quad \end{pmatrix} \end{array}$$

— i-я строка вычеркнута

$$\det\left(D_\varphi D_\varphi^T\right) = \Delta_1(x)^2 + \ldots + \Delta_n(x)^2.$$

$$N(x) = (\Delta_1, \Delta_2, \ldots, \Delta_n)$$

<u>Общая ситуация.</u>

$$\underbrace{\begin{pmatrix} B & \Big| \end{pmatrix}}_{n-1} \; n$$

$$B : \mathbb{R}^{n-1} \to \mathbb{R}^{n} \qquad Be_i$$

$$\begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{pmatrix} \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

$$\sum_j b_{ij} \Delta_j (-1)^{j} = 0$$

$$D = \begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1} & \frac{\partial \varphi_1}{\partial x_{n-1}} \\ \vdots & \\ \frac{\partial \varphi_n}{\partial x_1} & \frac{\partial \varphi_n}{\partial x_{n-1}} \end{pmatrix}$$

$$\varphi(x+\Delta) = \varphi(x) + \underline{D_x(\varphi)\cdot\Delta} + o(\Delta)$$

$\Delta = \sum \delta_k \vec{e}_k$ $\quad$ $\vec{e}_k$ - базис в $\mathbb{R}^{n-1}$

$$\varphi(x+\Delta) = \varphi(x) + \underline{\sum \delta_k D_x(\varphi)\vec{e}_k} + o(\Delta)$$

$D_x\varphi(e_k)$ - k-й столбец

$D_x(\varphi)$

$$\sum (-1)^{j+1} \Delta_j \cdot \vec{c} = 0$$

$\vec{c}$ – касательный вектор к Γ.

$$N(x) = \left((-1)^j \Delta_j(x)\right) \perp S$$

в точке x.

$$|N(x)|^2 = \sum_{j=1}^{m} \Delta_j^2(x)$$

$$\lambda_m(\varphi(E)) = \int_E |N(x)|^2 \, dx$$

Единичная нормаль к S: $\varphi(x)$

$$\nu(x) = \frac{N(x)}{|N(x)|}$$

Частный случай графика.

$U \subset \mathbb{R}^n \; ; \; f : U \to \mathbb{R}$

$\varphi : U \to \mathbb{R}^{n+1}$

$\varphi(x_1, \dots, x_n) \to$

$(f(x_1, \dots, x_n), x_1, \dots, x_n)$

$\int \sqrt{x_t^2 + y_t^2}\, dt$

$$D_\varphi = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \dots & \frac{\partial f}{\partial x_n} \\ 1 \quad 0 & \text{——} & 0 \\ 0 \quad 1 & \dots & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(-1)^{2+1}$$

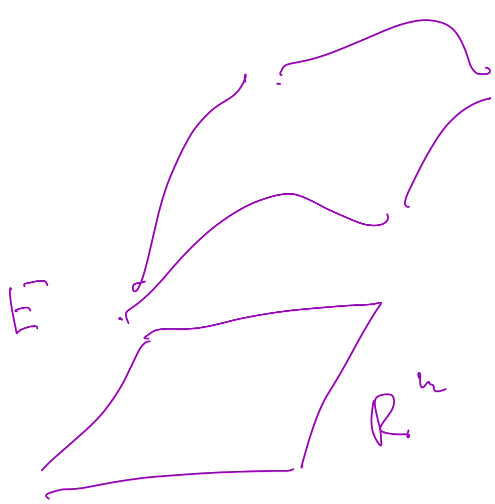
$$\Delta_1 = 1, \quad \Delta_2 = -\frac{\partial f}{\partial x_1}, \quad \Delta_3 = -\frac{\partial f}{\partial x_2}$$

$$\Delta_1 = 1; \quad \Delta_j = -\frac{\partial f}{\partial x_{j-1}}$$

$$N(x) = \left(1, -\frac{\partial f}{\partial x_1}, \dots, -\frac{\partial f}{\partial x_n}\right)$$

$$|N(x)| = \sqrt{1 + \left(\frac{\partial f}{\partial x_1}\right)^2 + \quad + \left(\frac{\partial f}{\partial x_n}\right)^2}$$

$$\int y(x)$$
$$\sqrt{1+y'^2}\,dt$$



$$\boxed{\lambda_n(\varphi(E)) = \int_E |N(x)|\, dx}$$

<u>Уточ.:</u> $f: \mathbb{R}^n \to \mathbb{R}$, $J(f) \neq 0$.

$S = \{(f(x), x),\ x \in U\}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cap$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathbb{R}^n$.

$\tilde{E} \subset S \Rightarrow$

$$\boxed{\lambda_n(\tilde{E}) = \int_{\varphi^{-1}(E)} |N(x)|\, dx}$$

$\chi_{\tilde{E}}$ – хар. ф-ція $\tilde{E}$ на S,

$$\int_S \chi_{\tilde{E}}\, d\lambda_{n-1} = \int_U \underbrace{\chi_{\tilde{E}}(\varphi(x))}_{\chi \circ \varphi} \overbrace{|N(x)|}^{dx}$$

Стандартний ищ. переход

$$\boxed{\int_S f\,d\lambda_n = \int_U (f\circ\varphi)(x)\,|N(x)|\,dx}$$

$$\sqrt{1+y'^2(t)}\,dt$$

$y \in S$ $\qquad f(y) = \dfrac{g(y)}{|N(\varphi^{-1}(y))|}$

$$\int_S g(y)\frac{1}{|N(\varphi^{-1}(y))|}\,d\lambda_n =$$

$$= \int_U g(\varphi(x))\,dx.$$

Это графика:

$$N(x) = \left(1, -\frac{\partial f}{\partial x_1}, \dots, -\frac{\partial f}{\partial x_n}\right)$$

$$\nu(x) = \frac{N(x)}{|N(x)|}$$

$$\frac{}{N(\varphi^{-1}(x))}$$ — компонента единичной нормали

Еще раз!

$$\int_S f(y)\,d\lambda_u = \int_U f(\underbrace{\varphi(x)}_{y})\,|N(x)|\,dx$$

$x \sim \varphi^{-1}(y)$

$$y = \varphi(x)$$

$$g(y) = f(y)\,|N(\varphi^{-1}(y))|$$

$$f(y) = \frac{g(y)}{|N(\varphi^{-1}(y))|}$$

$$= \int_U g(\varphi(x))\,dx$$

$$= \int_S \frac{g(y)}{|N(\varphi^{-1}(y))|}\, d\lambda_n$$

$$\int_S \frac{g(y)}{|N(\varphi^{-1}(y))|}\, d\lambda_n = \int_U g(\varphi(x))\, dx$$

Пусть Γ - график f

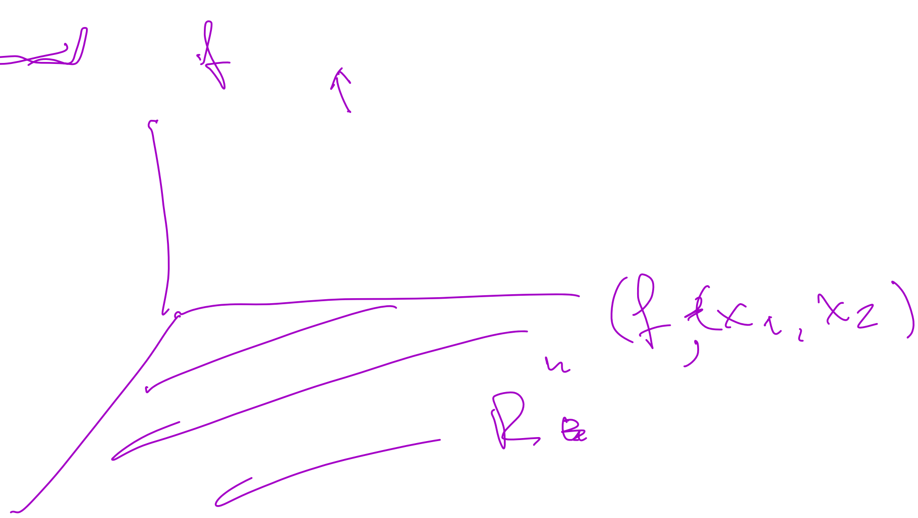
$$\varphi(x) = (f, x_1, \ldots, x_n)$$

$$N(x) = (1, -f_{x_1}, -f_{x_2}, \ldots, -f_{x_n})$$

нормаль!

Единичная нормаль:

$$\nu(x) = \frac{N(x)}{|N(x)|} = \left(\frac{1}{|N|}, -\frac{f_{x_1}}{|N|}, -\frac{f_{x_2}}{|N|}\right)$$



Формула Гаусса-Остроградского

Вернемся в размерность n.

Дано: $G \subset R^n$, ∂G - гладкая поверхность.

$\xi \in \partial G \Rightarrow \nu(\xi)$ — единичная внешняя нормаль к ∂G.

f — гладкая в окрестности G.

Тогда:

$$\int\limits_G \frac{\partial f}{\partial x_j}\, dx = \int\limits_{\partial G} f(y)\, \nu_j(y)\, dy.$$

$\nu = (\nu_1, \dots, \nu_n)$ — j-компонента единичной нормали.

$S \subset \mathbb{R}^n$ — гладкая поверхность если

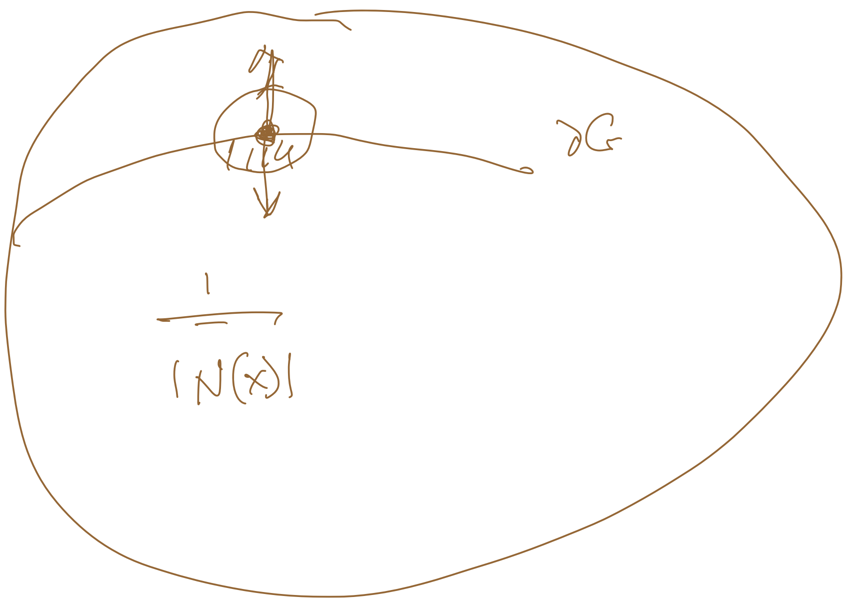
$\forall y \in S, \quad \exists V_y$ — окрестность

$$\varphi_y : \underset{\mathbb{R}^{n-1}}{\overset{\cap}{U}} \to V_y :$$

• $\varphi_y(x)$ – гладкая.

$$\operatorname{rank}\begin{pmatrix} \frac{\partial \varphi_{y,1}}{\partial x_1} & \dots & \frac{\partial \varphi_{y,1}}{\partial x_{n-1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \varphi_{y,n}}{\partial x_1} & & \frac{\partial \varphi_{y,n}}{\partial x_{n-1}} \end{pmatrix} = n-1$$

$n-1$

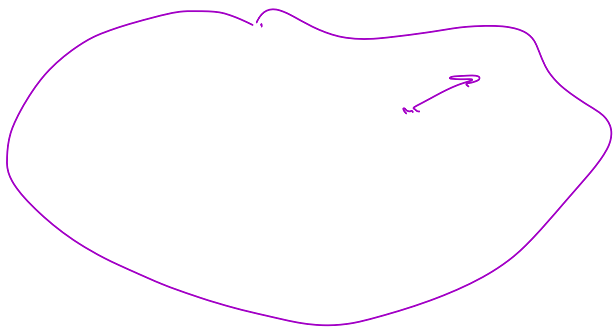


$$\frac{1}{|N(x)|}$$

1) Упражнение: $G = $ 

Докажите теорему.

2) $(f_1, \dots, f_n) = \vec{f}(x)$



$$\sum_j \int_G \frac{\partial f_j}{\partial x_j} dx = \int_{\partial G} f_j \nu_j \, dx$$

$$\int_G \sum_1^n \frac{\partial f_j}{\partial x_j} dx = \int_{\partial G} \sum_1^n f_j \nu_j \, dx$$

$$\operatorname{div} \vec{f} = \sum_{j=1}^n \frac{\partial f_j}{\partial x_j}$$

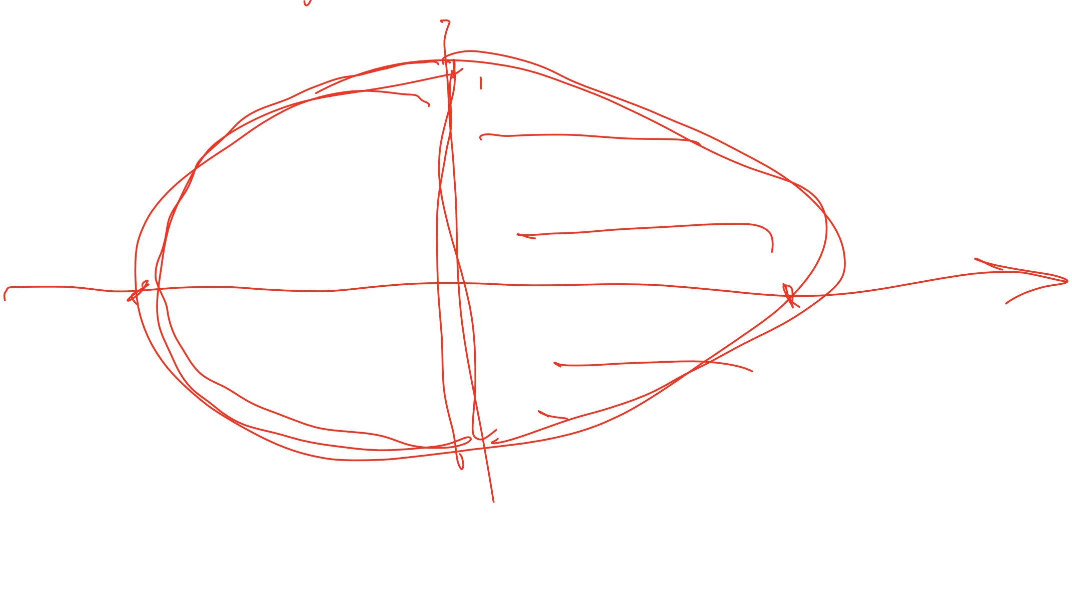
$$\sum f_j \nu_j = \langle \vec{f}, \vec{\nu} \rangle$$

$$\boxed{\int_G \operatorname{div} \vec{f}\, dx = \int_{\partial G} \langle \vec{f}, \vec{\nu} \rangle \, dy.}$$

$$\sum_j \int_G \frac{\partial f_j}{\partial x_j} dx = \int_{\partial G} f_j(y)\, \nu_j(y)\, dy.$$

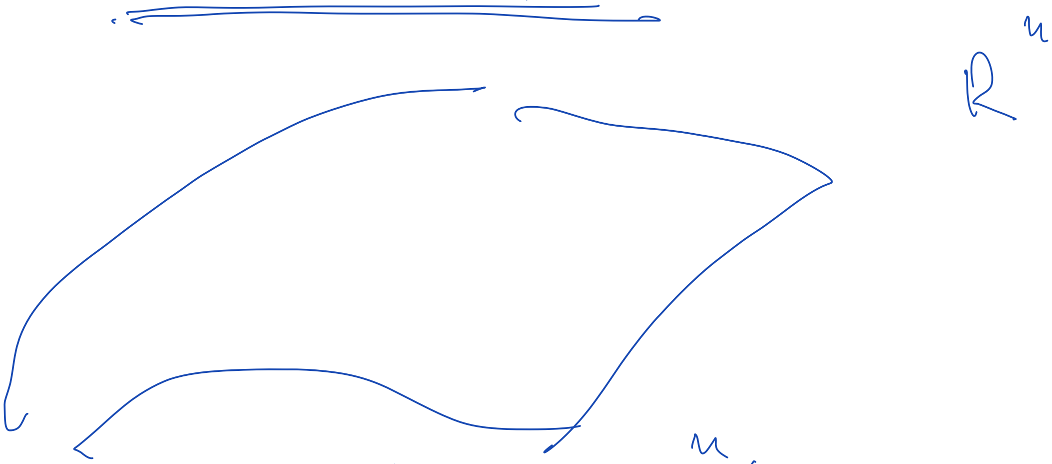
<u>Dazobe:</u>

1) Dok-bo.

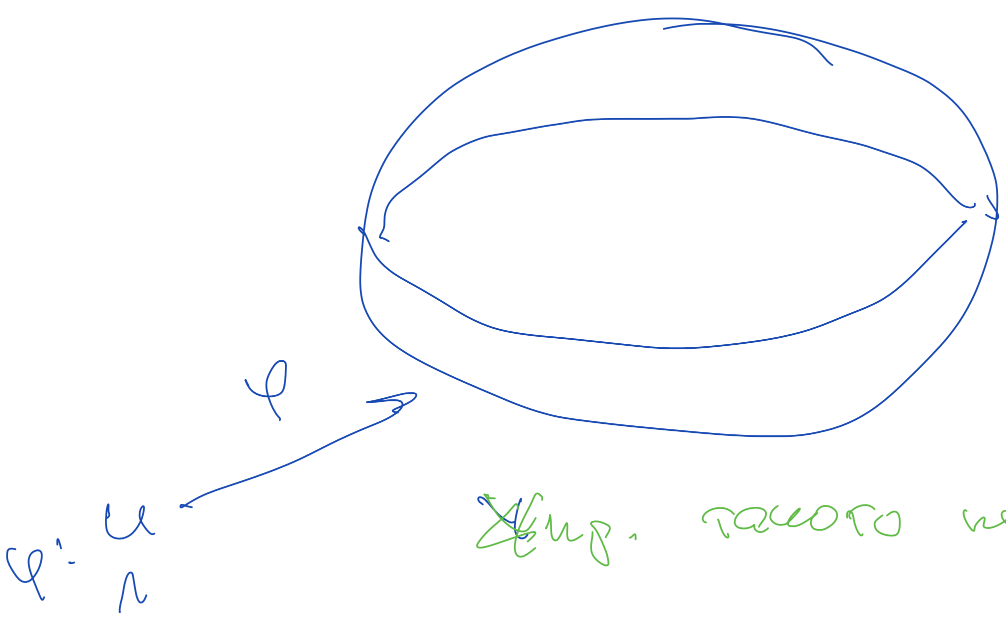


2). Многообразия.

Опред. многообразия:

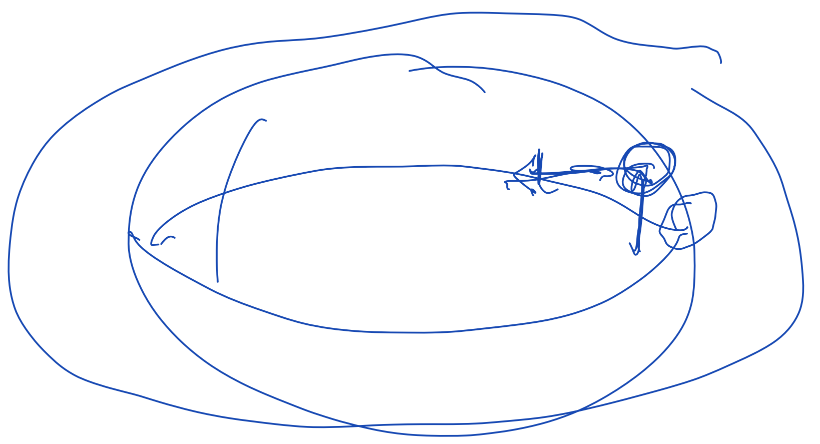


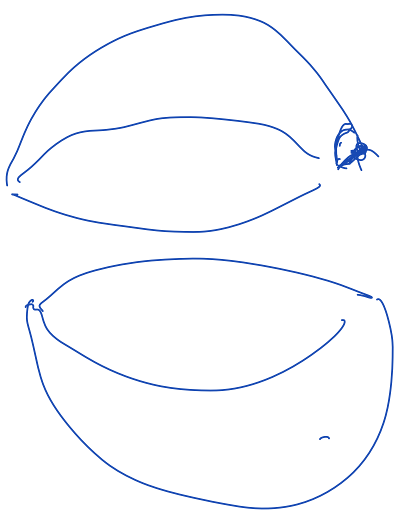
$\varphi : \mathbb{R}^m \to \mathbb{R}^n.$

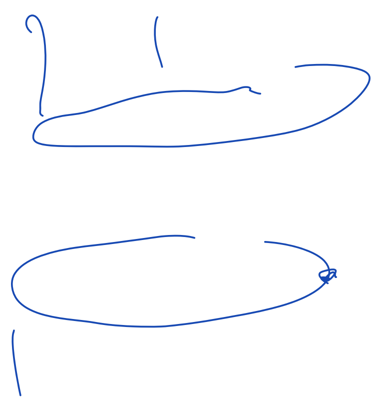


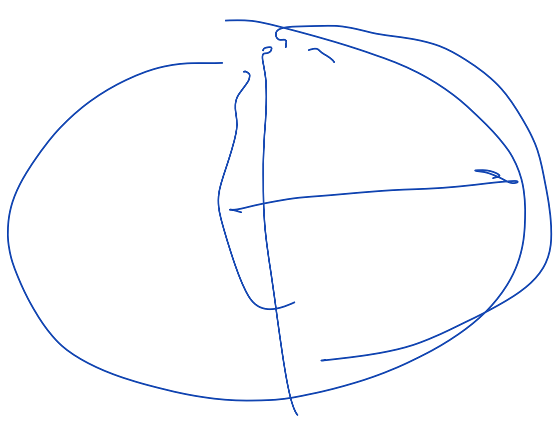
Пр. такого не $\exists$.

R^2









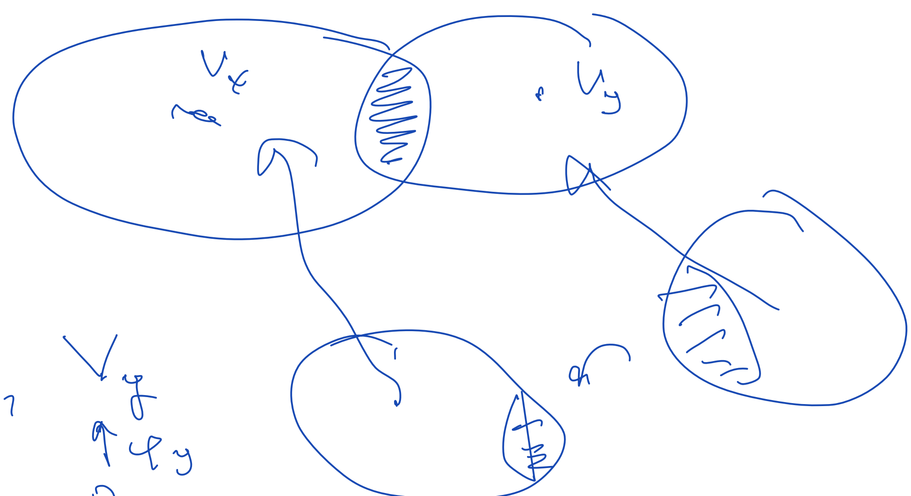
Определение гладкого многообр. размерности n.

Тополог. ипр-во X такое что.

1) $\forall x \in X \quad \exists V_x$ - окр. x

и ~~отобр.~~ биекция $\varphi_x : B \to V_x$
шар в R^n.

2)



$V_x, \quad V_y$

$\varphi_x \uparrow \quad \uparrow \varphi_y$

$B_x \quad B_y$

$U_x = \varphi_x^{-1}(V_x \cap V_y) \subset B_x$

$U_y = \varphi_y(V_x \cap V_y) \subset B_y$

$(\varphi_y^{-1}, \varphi_x)(U_x) \sim \quad \varphi_y^{-1}\varphi_x : U_x \to U_y.$

гладко.

$\varphi_y^{-1}\varphi_x \in C^k.$