# WEIGHTED INEQUALITIES FOR THE TWO-DIMENSIONAL HARDY OPERATOR

## (I) THE STORY

This here is an attempt to tell a certain story /taken out from a /still/ongoing research/, as it can be seen from a certain specific point of view.

The idea is that we introduce a certain **MODEL**, which, as it turns (hopefully) out, can serve as a way to handle a bunch of somewhat related phenomena. These relations, I believe, are known, though rather in the usual "folklore" sense.

For this talk here I've chosen a sort of an 'upwards' approach, that is we are mainly talking about a very particular case of the **MODEL**, which is still plenty interesting in its own right (and also has at least two open problems connected to it.

## (II) THE MODEL

It is rather simple.

So, consider a finite rooted directed graph $\Gamma$ without directed cycles — — same as a finite partially ordered set with a unique maximal element.

The order relation is denoted by "$\leq$" or "$\subseteq$" or whatever else fits.

As in $\alpha \leq \beta$ means that both $\alpha$ & $\beta$ are vertices of $\Gamma$/elements of the aforementioned set, and that also $\alpha$ is 'below' (my graphs grow downwards) than $\beta$.

So, going back to the model: Everything is FINITE.

The main object is a collection of non-negative numbers attached to vertices of $\Gamma$, namely a map $\Gamma \to \mathbb{R}_+$. Depending on the context we call it function, weight or a measure.

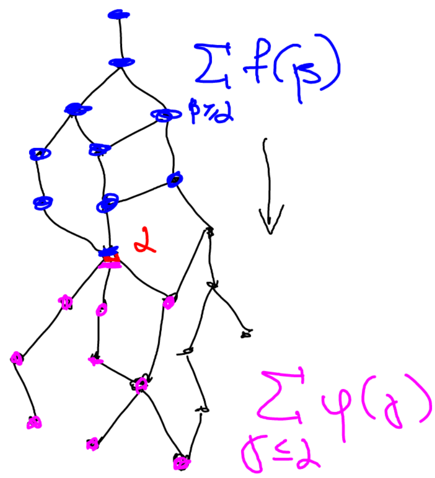
## Hardy operator

We define the Hardy operator (unweighted) $I$ as follows:

$$(If)(\alpha) := \sum_{\beta \geq \alpha} f(\beta), \quad f: \Gamma \to \mathbb{R}_+$$

and its 'adjoint' $I^*$ by

$$(I^*\varphi)(\alpha) := \sum_{\gamma \leq \alpha} \varphi(\gamma), \quad \varphi: \Gamma \to \mathbb{R}_+.$$




## The problem

Given a pair of weights $\mu, \sigma: \Gamma \to \mathbb{R}_+$, we ask when — — as in under which conditions on $\sigma, \mu$ — the Hardy operator is bounded acting from $L^2(\Gamma, \sigma)$ to $\ell^2(\Gamma, \mu)$,

$$\sum_{\alpha \in \Gamma} (If)^2(\alpha)\mu(\alpha) \leq C \cdot \sum_{\alpha \in \Gamma} f^2(\alpha)\sigma(\alpha)$$

for any $f$. Taking $f\sigma^{-1}$ instead of $f$ we obtain a slightly different form of the same embedding

$$\sum_{\alpha \in \Gamma} (I_w f)^2(\alpha)\mu(\alpha) \leq C \cdot \sum_{\alpha \in \Gamma} f^2(\alpha)\omega(\alpha),$$

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where $\omega = \sigma^{-1}$, and $(I_\omega f)(\alpha) = (I(\omega \cdot f))(\alpha)$.

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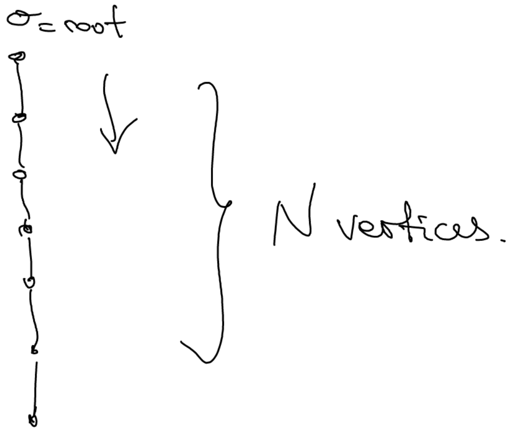
Generally speaking there are no (and couldn't be any) conditions on $(\mu, \omega)$ that provide $L^2$-boundedness of the embedding without specifying the nature of the graph $\Gamma$. Henceforth examples.

## Examples.

$\boxed{A}$ $\Gamma$ is just a branch of some unspecified depth $N$:




The condition B then:

$$\sup_{\alpha \in \Gamma} (I^* \mu)(\alpha) \cdot (I\omega)(\alpha) \leq \tilde{C}, \text{ for some constant } \tilde{C},$$

which is proportional to $C$ from (53). Of course no dependence on $N$!

This was proven in '60-'70s by Tomaselli, Talenti, Muckenhoupt.

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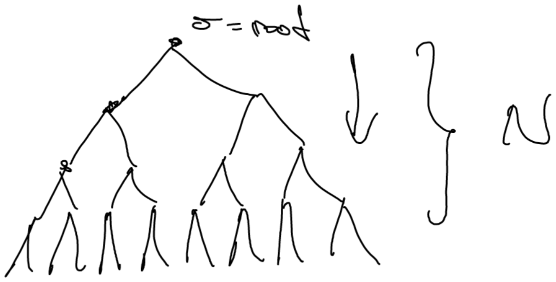
$\boxed{B}$ Now $\Gamma$ is a finite rooted dyadic tree of depth $N$.




There are several equivalent conditions here:

Multiple box

$\boxed{B1}$ $$\sum_{\alpha \in E} (I^* \mu)^2(\alpha)\, \omega(\alpha) \leq C \cdot \sum_{\alpha \in E} \mu(\alpha), \text{ for any downset}$$

$E$ (downset means that if $\alpha \in E$, then $\beta \in E$ $\forall \beta \leq \alpha$).

**B2** $\quad \sum_{\alpha \le \gamma} (I^*\mu)^2(\alpha)\, w(\alpha) \le C \cdot (I^*\mu)(\gamma)$ for any $\gamma \in \Gamma$.

single box

Clearly this condition is strictly weaker than $\boxed{B1}$.

**B3** $\quad \mu(E) \le C \cdot \mathrm{Cap}_w(E)$ for any downset $E$.

subcapacity condition

The capacity $\mathrm{Cap}_w$ is defined as follows:

$$\mathrm{Cap}_w(E) := \inf_f \left\{ \|f\|^2_{L^2(\Gamma, w)} : I_w f \ge 1 \text{ on } E \right\}.$$

There is no single box version of this one!

Courtesy of Nazarov-Treil-Volberg & Arcozzi-Rochberg-Sawyer-Wick.

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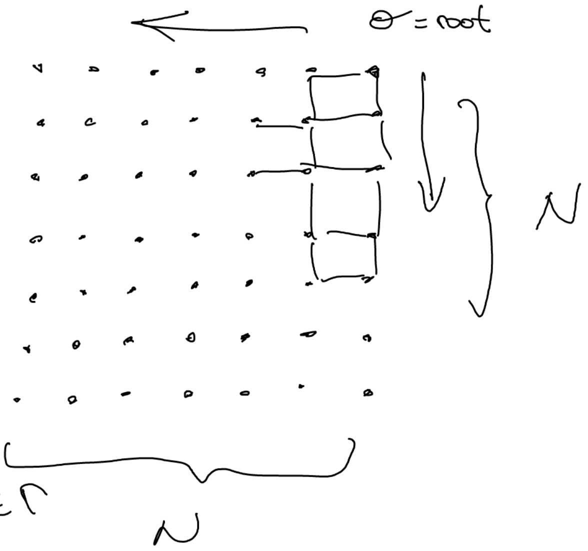
**C** $\quad$ And now $\Gamma$ is $\Gamma_1 \times \Gamma_2$ with both coordinate graphs like in example $\boxed{A}$.




The conditions worsen: there are three single boxes to check

- $\sup_\alpha (I w)(\alpha)\,(I^*\mu)(\alpha) \le C$
- $\sum_{\alpha \le \gamma} (I^*\mu)^2(\alpha)\, w(\alpha) \le C \cdot I^*\mu(\gamma) \quad \forall \gamma \in \Gamma$
- $\sum_{\alpha \ge \gamma} (I w)^2(\alpha)\, \mu(\alpha) \le C \cdot I w(\gamma) \quad \forall \gamma \in \Gamma$.

All of them are necessary.

E. Sawyer showed this (actually for $L^p \to L^q$ embedding).

$\boxed{D}$ And now $\Gamma = \bigotimes_{k=1}^{n} \Gamma_k$ with $\Gamma_k$ being dyadic trees.

We have come to the extent of our current knowledge here.

$\boxed{D21}$ For $n=2$, $w=1$ the subcapacitory condition should hold:

- $\mu(E) \le C \cdot \mathrm{Cap}(E) \quad \forall E$-downset

$\boxed{D22}$ For $n=3$, $w=w_1 \cdot w_2 \cdot w_3$ one single box is enough

- $$\sum_{\alpha \le \gamma} (I^* \mu)^2 (\alpha) w(\alpha) \le C \cdot (I^* \mu)(\gamma) \quad \forall \gamma.$$

And that's it! Nothing is known for general weights $w$ or for $n \geq 4$.

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## SOME PROOFS (SAWYER RELATED)

We are going to explain how to handle cases $\boxed{A}$ and $\boxed{C}$. This (in slightly different context) was shown by Eric Sawyer in his

*Weighted inequalities for the two-dimensional Hardy operator*, STUDIA MATHEMATICA, v. 72, 1985

paper. We will also be making some detours into potential-theoretic side of things and connections to the general MODEL.

CASE $\boxed{A}$ We start by reintroducing the problem in the 'continuous' language - the one Sawyer actually used.

## Continuous setting

Consider the $n$-dimensional Hardy operator $I_n$ and its adjoint $I_n^*$, that are given by

$$(I_n f)(x_1,\dots,x_n) = \int_0^{x_1}\dots\int_0^{x_n} f(t_1,\dots,t_n)\,dt_1\dots dt_n$$

$$(I_n^* f)(x_1,\dots,x_n) = \int_{x_1}^{\infty}\dots\int_{x_n}^{\infty} f(t_1,\dots,t_n)\,dt_1\dots dt_n,$$

Where $f$ is some integrable function on $(\mathbb{R}_+)^n$.

Given a pair of non-negative weight functions $\mu, v : (\mathbb{R}_+)^n \to \mathbb{R}_+$ we ask when

$$\left(\int_{(\mathbb{R}_+)^n} |I_n f|^q \mu(x)\,dx\right)^{\frac{1}{q}} \le C\cdot\left(\int_{(\mathbb{R}_+)^n} |f(x)|^p v(x)\right)^{\frac{1}{p}} \quad \boxed{S52}$$

for any $f$. Here, naturally, $C = C(p,q,v,\mu)$. Also it is immediate that one has to check (S52) only for continuous non-negative functions $f$. Also also, taking $fv^{-\frac{1}{p-1}}$ instead of $f$ one gets

$$\left(\int_{(\mathbb{R}_+)^n} |I_n(fw)|^q(x)\,\mu(x)\,dx\right)^{\frac{1}{q}} \le C\left(\int_{(\mathbb{R}_+)^n} |f(x)|^p w(x)\right)^{\frac{1}{p}}, \quad \boxed{S53}$$

where $w = v^{\frac{1}{1-p}}$.

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Talenti, Tomasselli, Muckenhoupt showed that for $n=1$ (S53) holds if and only if

$$\sup_{x\in\mathbb{R}_+} (I_1^*\mu)^{\frac{1}{q}}(x)\,(I_1 w)^{\frac{1}{p'}}(x) = A < +\infty$$

for some absolute constant $A$, and that $A \sim C$ from (S53).

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This naturally corresponds to the scenario $\boxed{A}$ in our model. The difference is that

- here we cover different possible values of $p$ and $q$ — unlike $\boxed{A}$, where $p = q = 2$
- here everything is continuous, and also not necessarily finite, unlike $\boxed{A}$

Both of these differences we happily disregard. The first one is due to the fact that even the 'linear' case $p = q = 2$ is already complicated enough and higlight the main ideas. The second one is essentially unimportant, technical in nature, and the details are left as an exercise.

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So how do we prove that the condition $\boxed{A}$ is actually equivalent to the $L^2$-boundedness of the embedding (53)?

The usual way — one direction, (53) $\Longrightarrow$ condition, goes via testing (53) on some appropriate function $f$. The choice of such a function is fairly obvious once the condition is known, and in this particular instance we postpone it for a time being.

The opposite direction, that is condition $\Rightarrow$ (53), is usually way harder, and by now there are several different approaches. Here we shall use the one related to the Potential Theory. It is not the most straightforward and may seem overly complicated (and we won't use it for $\boxed{C}$), but it does shed some light on the interplay between combinatorics and potentials, moreover this simple scenario looks just right for introducing this technique.

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So with that in mind we let $\Gamma$ to be a one-branch graph of finite depth (integers from 1 to $N$ in other words), and we fix $\mu, w: \Gamma \to \mathbb{R}_+$.

We also define for $E \subset \Gamma$

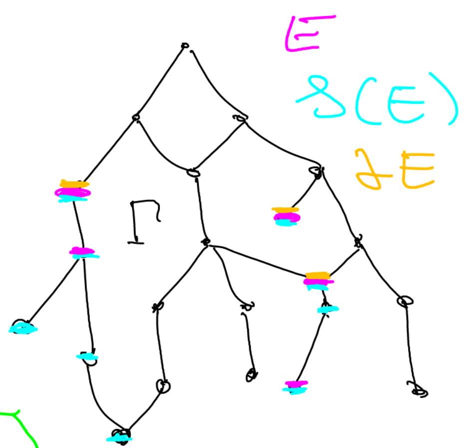
$$\operatorname{Cap}_w E := \inf \{ \|f\|^2_{L^2(\Gamma, dw)} : I_w f \geq 1 \text{ on } E \}.$$

Note that this definition works for any $\Gamma$ – directed without directed cycles.

Moreover, by general theory (which we will not bother to reproduce here) we have

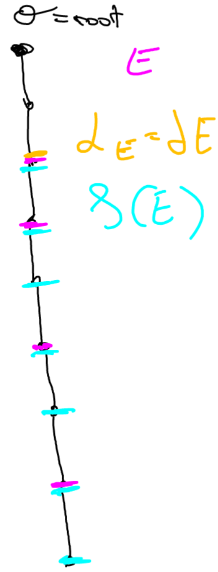
- there exists a unique minimizer $f_E$
- actually, $f_E = I^* \mu_E$ for some $\mu_E : \Gamma \to \mathbb{R}_+$, we call this $\mu_E$ the equilibrium measure of $E$
- $I f_E = I_w I^* \mu_E =: V_w^{\mu_E} = 1$ quasi almost everywhere on $E$, that is everywhere, but on a set of zero capacity (this is Frostman's theorem)
- every singleton $\{\alpha\}$, with $\alpha \in \Gamma$, has strictly positive capacity (this is due to the fact that $\Gamma$ is finite graph, and $w$ takes finite values), in particular it follows that 'quasi almost everywhere' is just 'everywhere' in our scenario – it does make life much easier.
- clearly, both $f_E, \mu_E$ are positive
- if $\alpha \in E$ and $\beta \leq \alpha$ (i.e. $\beta$ lies 'below' $\alpha$ on $\Gamma$), then $(I_w f)(\beta) \geq (I_w f)(\alpha)$ for $f \geq 0$, hence, in particular, $I_w f_E (\beta) \geq 1$ for any $\beta \leq \alpha$ s.t. $\alpha \in E$, and $\mathrm{Cap}_w(E) = \mathrm{Cap}_w(\mathcal{S}(E))$, where $\mathcal{S}(E)$ is the successor set of $E$, $\mathcal{S}(E) = \{\beta \leq \alpha : \alpha \in E\}$
- on the other hand, by the same reasoning, $\mathrm{Cap}_w(E) = \mathrm{Cap}_w(\partial E)$, where $\partial E$ is the (upper) boundary of $E$,

$$\partial E = \{\alpha \in E : \text{if } \beta > \alpha, \text{ then } \beta \notin E\}$$




Going back now to our particular case $\Gamma$ = 'single branch', we see that capacity is easy to compute (generally it is highly unusual).

Namely, if $E \subset \Gamma$, then there must be a unique maximal element $\alpha_E \in E$, so that $\partial E = \{\alpha_E\}$, and




$$\mathrm{Cap}_\omega E = \mathrm{Cap}_\omega \{\alpha_E\} = \mathrm{Cap}_\omega S(E).$$

Now by the powers of calculus (see 'extrema of functions'),

$$\inf \{ \|f\|_{L^2(\Gamma,\omega)} : I_\omega f \geq 1 \text{ on } E \} =$$

$$= \inf \Big\{ \sum_{\alpha \in \Gamma}' f^2(\alpha)\,\omega(\alpha) : \sum_{\beta \geq \alpha_E}' f(\beta)\,\omega(\beta) \geq 1 \Big\} \circleddash$$

$$\boxed{\circleddash \; \frac{1}{\sum\limits_{\beta \geq \alpha_E}' \omega(\beta)} = \frac{1}{(I\omega)(\alpha_E)}.}$$

This means that the condition $\boxed{A}$, which is

$$\sup_{\alpha \in \Gamma} (I\omega)(\alpha)\,(I^*\mu)(\alpha) = A < +\infty$$

is actually equivalent to

$$\sup_{\alpha \in \Gamma} \frac{1}{\mathrm{Cap}_\omega(\alpha)} (I^*\mu)(\alpha) = A,$$

or, in other words,

$$\sup_{\alpha \in \Gamma} \frac{\mu(S(\alpha))}{\mathrm{Cap}_\omega(\alpha)} = A, \text{ where } \mu(S(\alpha)) = \sum_{\beta \in S(\alpha)}' \mu(\beta) =$$

$$= \sum_{\beta \leq \alpha}' \mu(\beta) = (I^*\mu)(\alpha).$$

We can further reformulate it to be a subcapacitary condition,

$$\boxed{\mathrm{Cap}_\omega(E) \leq A \cdot \mu(E) \quad \forall E \subset \Gamma.} \qquad (SC71)$$

We will see this one later on.

So, it might seem that we are overcomplicating things, but now we can actually use the following scheme (probably invented by V. Maz'ya). Namely, to estimate the LHS of (53) we write

$$\sum_{\alpha\in\Gamma} (I_w f)^2(\alpha)\mu(\alpha) \sim \sum_{k\in\mathbb{Z}} 3^{2k}\cdot\mu(\{\alpha : I_w f \geq 3^k\}).$$

Next we use our subcapacitary condition to realize that

$$\sum_{k\in\mathbb{Z}} 3^{2k}\mu(\{\alpha : I_w f \geq 3^k\}) \leq A\cdot \sum_{k\in\mathbb{Z}} 3^{2k}\,\mathrm{Cap}_w(\{\alpha : I_w f \geq 3^k\}).$$

The final step is the application of the so-called strong capacitary inequality

$$\sum_{k\in\mathbb{Z}} \lambda^{2k}\cdot \mathrm{Cap}_w(\{\alpha : I_w f \geq \lambda^k\}) \leq C_\lambda \cdot \|f\|^2_{L^2(\Gamma,w)}, \quad \lambda>1.$$

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This inequality, SCI as we call it, is a natural modification of the trivial weak capacitary inequality, which is

$$\mathrm{Cap}_w(\{\alpha : I_w f \geq \lambda\}) \leq \frac{\|f\|^2_{L^2(\Gamma,w)}}{\lambda^2}, \quad \lambda>1.$$

The proof proceeds just by examining the definition of capacity, and this weak capacitary inequality always holds (any $\Gamma, w$).

On the other hand, SCI does not necessarily have to be true (and we will see that later).

Nevertheless, for some choice of $\Gamma$ and $w$, in particular for this one, it does indeed hold true. So how do we see it?

Fix a function $f \geq 0$, and consider $\Omega_k := \{\alpha : I_w f \geq 3^k\}$.
Next assume that $(I_w f)(\alpha_k) = 3^k$, where $\alpha_k = \alpha_{\Omega_k}$ is the maximal element in $\Omega_k$.
It will take some effort to remove this assumption, but the work to be done is entirely technical, and can be left as an exercize.

One can easily see now that

$$\sum_{\alpha_k > \alpha \geq \alpha_{k+1}} f(\alpha) w(\alpha) = 3^{k+1} - 3^k \sim 3^k.$$

In particular if we define $f_k := f \cdot \mathbb{1}_{\Omega_{k+1} \setminus \Omega_k}$,
then $I_w f_k \gtrsim 3^{k+1}$ on $\Omega_{k+1}$, hence

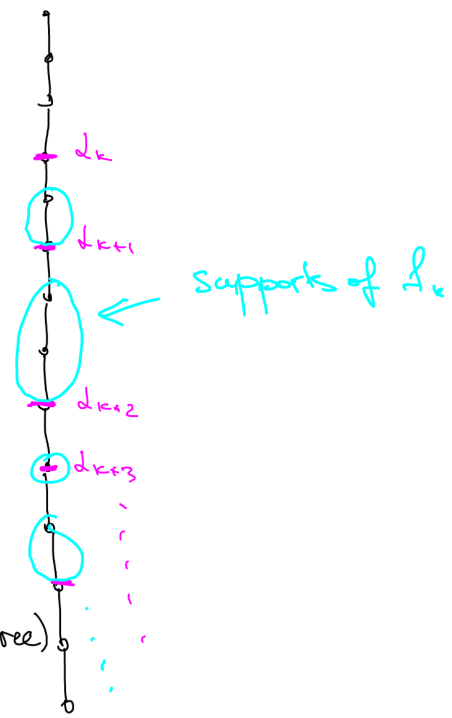
$$\mathrm{Cap}_w \Omega_{k+1} \lesssim 3^{-2k} \cdot \| f_k \|^2_{L^2(\Gamma, w)}.$$

But since the supports of $f_k$ are disjoint, we immediately have

$$\sum_k 3^{2k} \mathrm{Cap}_w \Omega_k \lesssim \sum_k \| f_k \|^2_{L^2(\Gamma, w)} \sim \Big\| \sum_k f_k \Big\|^2_{L^2(\Gamma, w)} \sim$$

$$\sim \| f \|^2_{L^2(\Gamma, w)},$$

and we are done.




The key ingredient here is the lack of cycles, i.e. all geodesics are unique.
This is also true for the tree (as in $\boxed{B}$), but not for $\mathbb{Z}^2$ (case $\boxed{C}$). Sometimes we can get around this obstacle ($w \equiv 1$, $\Gamma$ – bitree), sometimes we can not (general $w$, $\Gamma = \mathbb{Z}^2$).

The other direction here is obvious — by now it is clear what should we test (53) on. Indeed, fix any $\alpha \in T$, and consider such $f$, that $(I_\omega f)(\alpha) \geq 1$. Then

$$(I^*\mu)(\alpha) = \sum_{\beta \leq \alpha} \mu(\beta) \leq \sum_{\beta \leq \alpha}{}' (I_\omega f)^2(\beta)\mu(\alpha) \overset{\text{by (53)}}{\leq} C \cdot \sum_{\alpha}{}' f^2(\alpha)\omega(\alpha) =$$

$$= C \cdot \|f\|^2_{L^2(\Gamma,\omega)}.$$

Minimizing over all such $f$ we obtain $\mathrm{Cap}_\omega(\{\alpha\})$ on the right-hand side, which means that

$\mu(S(\alpha)) \lesssim \mathrm{Cap}_\omega\{\alpha\}$, and we have our subcapacitary condition. ⊠

And now we move to $\Gamma = \Gamma_1 \times \Gamma_2$ — cartesian product of two single branches (that models $\mathbb{Z}^2$). The argument above does not work anymore (no SCI in general). Maybe it can be modified after all (and we will collect some supporting evidence), but we don't know how to do it yet.

We start with the same idea — decompose the LHS of (53) via the distribution function, namely

$$\sum_{\alpha \in \Gamma}{}' (I_\omega f)^2(\alpha)\mu(\alpha) \sim \sum_{k \in \mathbb{Z}} 3^{2k}\mu(\Omega_{k+1} \setminus \Omega_k), \text{ where}$$

$$\Omega_k = \{\alpha : (I_\omega f)(\alpha) \geq 3^k\}.$$

Let us look at these sets (see the picture below). Again, as before, we assume (claiming there is no loss of generality here and omitting the proof of this claim), that the boundary sets $\partial\Omega_k$ consist of exactly those points where $I_\omega f = 3^k$. ⊛

In other words, $\partial\Omega_k$ are 'time-stop' curves — only now we have two-dimensional time.

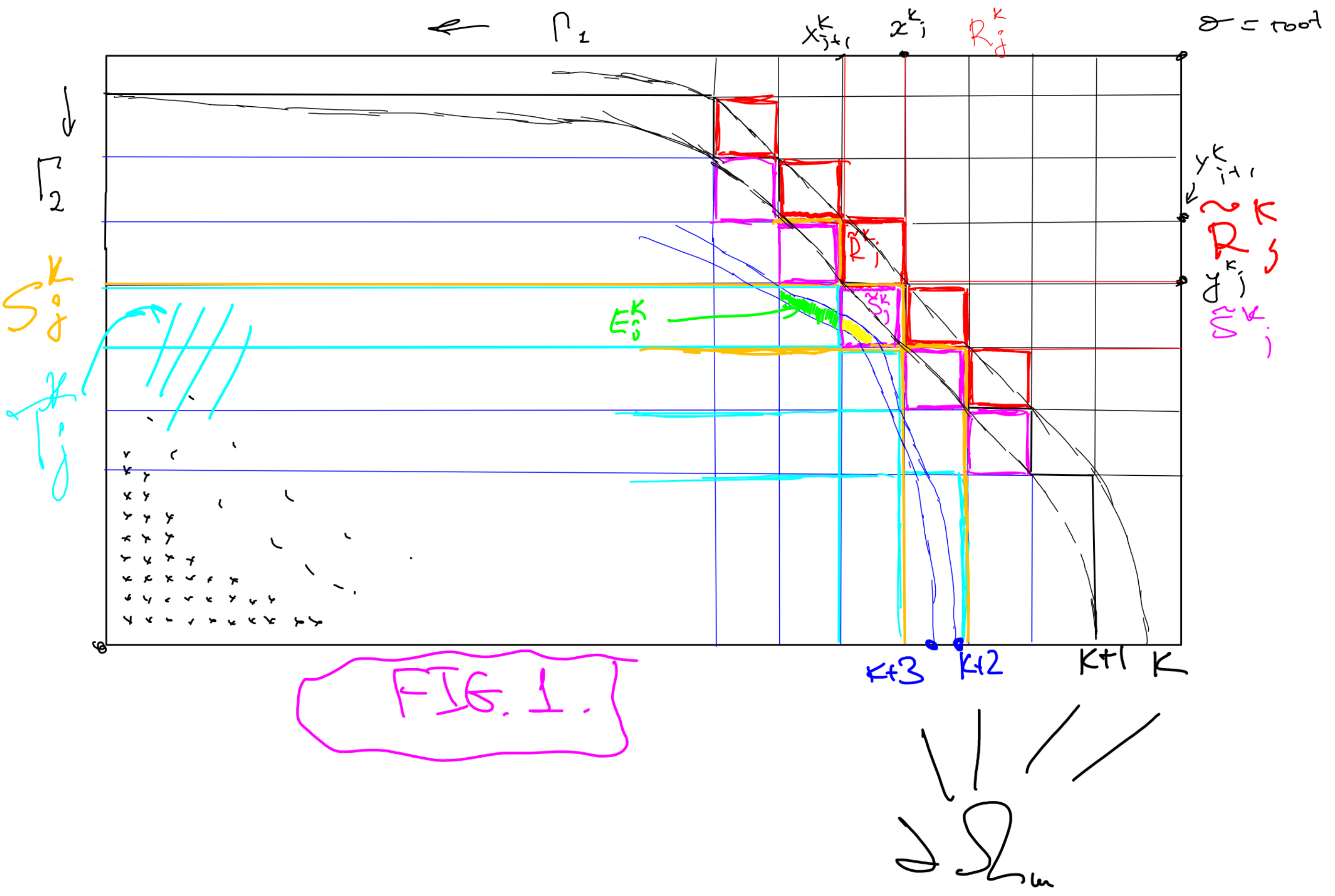



FIG. 1.

So, this is the main picture, which is made out of the following pieces:

- we fix some number $k \in \mathbb{Z}$, and consider the domains $\Omega_k, \Omega_{k+1}, \Omega_{k+2}, \Omega_{k+3}$, and also their boundaries $\partial\Omega_k, \partial\Omega_{k+1}, \partial\Omega_{k+2}, \partial\Omega_{k+3} \underset{\text{by our assumption (*) above}}{=} \{\alpha : I_\alpha f = 3^i,\ i = k \dots k+3\}$. These are drawn as curves, blue for curves $k+2, k+3$ and black for $k+1, k$. Naturally $(f \neq 0)$ they are monotone as well and lie strictly below each other.

- Next we construct a staircase between $\partial\Omega_k$ and $\partial\Omega_{k+1}$, namely we consider a collection of points $\alpha_j^k = (x_j^k, y_j^k)$, where $\alpha_j^k \in \partial\Omega_k$, $x_j^k, y_j^k$ are its coordinates on $\Gamma_1$ and $\Gamma_2$ respectively, and $(x_j^k, y_{j-1}^k) \in \partial\Omega_{k+1}$. See FIG 1 above.

- After cooking up a staircase, we define a bunch of rectangles: $S_j^k, \tilde{S}_j^k, R_j^k, \tilde{R}_j^k, T_j^k$ as follows:

$\leadsto S_j^k = \mathcal{S}(\alpha_j^k) = \mathcal{S}((x_j^k, y_j^k)) =$

$= \{\beta \in \Gamma_1 \times \Gamma_2 : \beta \leq \alpha_j^k\}$ – a rectangle with upper right corner at $\alpha_j^k$, or, which is the same, the successor set of $\alpha_j^k$.

$$\leadsto \tilde{S}_j^k = \{\beta = (x,y) : x_{j+1}^k \leq x \leq x_j^k ;\ y_{j-1}^k \leq y \leq y_j^k\} =$$

$$= S_j^k \setminus (T_j^k \cup T_{j-1}^k)$$

$$\leadsto R_j^k = \mathcal{P}((x_{j+1}^k, y_j^k)) = \left\{\beta = (x,y) \in \Gamma_1 \times \Gamma_2 : \begin{array}{l} x \geq x_{j+1}^k \\ y \geq y_j^k \end{array}\right\}$$

$$\leadsto \tilde{R}_j^k = \{\beta = (x,y) : x_{j+1}^k \leq x \leq x_j^k ;\ y_j^k \leq y \leq y_{j+1}^k\}$$

$$\leadsto T_j^k = \{\beta = (x,y) : x \leq x_{j+1}^k ;\ y \leq y_j^k\}$$

- Finally we consider the set (between two curves $\partial\Omega_{k+2}$ and $\partial\Omega_{k+3}$) $\Omega_{k+2} \setminus \Omega_{k+3}$, and split it into two parts:

YELLOW: $F_j^k = \Omega_{k+2} \setminus \Omega_{k+3} \cap \tilde{S}_j^k$

and

Green: $E_j^k = \Omega_{k+2} \setminus \Omega_{k+3} \cap T_j^k$.

Clearly, $\Omega_{k+2} \setminus \Omega_{k+3} = \left(\bigcup_j F_j^k\right) \cup \left(\bigcup_j E_j^k\right)$,

and all these sets are disjoint.

Now we take the LHS of **(53)** and start working with it.

We have:

$$\sum_{\alpha \in \Gamma} (I_w f)^2(\alpha)\mu(\alpha) \sim \sum_{k \in \mathbb{Z}} 3^{2k} \mu\{\alpha \in \Gamma : 3^{k+2} \leq I_w f(\alpha) \leq 3^{k+3}\} =$$

$$= \sum_{k \in \mathbb{Z}} 3^{2k} \mu(\Omega_{k+2} \setminus \Omega_{k+3}) = \sum_{k \in \mathbb{Z}} 3^{2k} \mu\left(\bigcup_j F_j^k\right) + \sum_{k \in \mathbb{Z}} 3^{2k} \mu(E_j^k) =$$

$$= \mathrm{I} + \mathrm{II}.$$

We start with estimating part (I). Let us recall the three conditions:

(CS1) $\sup_{\gamma\in\Gamma} (I^*\mu)(\gamma)\,(Iw)(\gamma) \le A^2$;

(CS2) $\sum'_{\alpha\le\gamma} (I^*\mu)^2(\alpha)\, w(\alpha) \le A^2 \cdot (I^*\mu)(\gamma), \quad \forall\gamma\in\Gamma$

(CS3) $\sum'_{\alpha\ge\gamma} (Iw)^2(\alpha)\,\mu(\alpha) \le A^2\cdot (Iw)(\alpha), \quad \forall \gamma\in\Gamma$

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To estimate part (I) we actually need only one of these conditions – (CS2). We proceed by observing that since

$$(I_w f)(x^k_{j+1}, y^k_j) = I_w(x^k_j, y^k_{j-1}) = 3^{k+1} \text{ and}$$

$$(I_w f)(\beta) \ge 3^{k+2} \text{ for } \beta\in\Omega_{k+3}\setminus\Omega_{k+2} \text{ and}$$

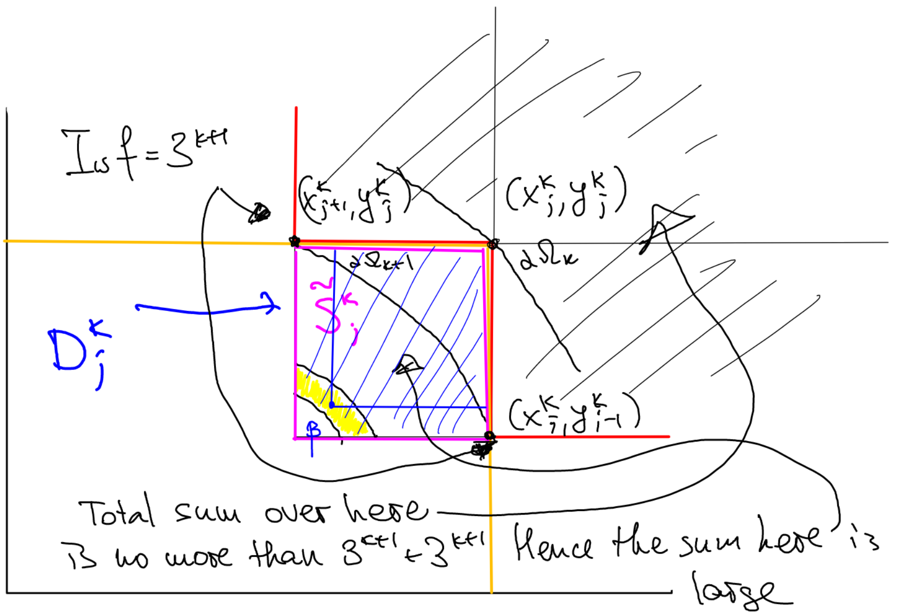
$$(I_w f)(\beta) = \sum'_{\alpha\ge\beta} f(\beta)w(\beta) \le \sum'_{\alpha\ge(x^k_{j+1},y^k_j)} f(\beta)w(\beta) + \sum'_{\alpha\ge(x^k_j,y^k_{j-1})} f(\beta)w(\beta) + \sum'_{\beta\le\alpha\le(x^k_j,y^k_j)} f(\beta)w(\beta),$$ we see that

$$\sum'_{\beta\le\alpha\le(x^k_j,y^k_j)} f(\beta)w(\beta) \ge 3^{k+2}-3^{k+1}-3^{k+1}=3^{k+1},$$ see Fig below

So let $D^k_j := \tilde S^k_j \setminus \Omega_{k+3}$.
It follows immediately that

$$(I_w(f\cdot 1_{D^k_j}))(\beta)\ge 3^{k+1}$$

for $\beta\in\tilde S^k_j\cap(\Omega_{k+2}\setminus\Omega_{k+3}) =$
$= F^k_j$.



Thus we may write

$$\mu(F_j^k) \le 3^{-k} \sum_{\alpha \in F_j^k} \left(I_w(f \cdot \mathbb{1}_{D_j^k})\right)(\alpha)\,\mu(\alpha) =$$

$$= 3^{-k} \sum_{\alpha \in F_j^k} \Bigg( \sum_{\substack{\gamma \ge \alpha \\ \gamma \in D_j^k}} f(\gamma) w(\gamma) \Bigg) \mu(\alpha) = 3^{-k} \sum_{\gamma \in D_j^k} f(\gamma) w(\gamma) \sum_{\substack{\alpha \le \gamma \\ \alpha \in F_j^k}} \mu(\alpha) \le$$

$$\le 3^{-k} \cdot \sum_{\gamma \in D_j^k} f(\gamma) w(\gamma) \cdot (I^*\mu)(\gamma) \le \text{Hölder inequality} \le$$

$$\le 3^{-k} \Bigg( \sum_{\gamma \in D_j^k} f^2(\gamma) w(\gamma) \Bigg)^{\frac{1}{2}} \Bigg( \sum_{\gamma \in D_j^k} (I^*\mu)^2(\gamma) w(\gamma) \Bigg)^{1/2} \le$$

$$\overset{\text{since } D_j^k \subset S_j^k}{\le} 3^{-k} \Bigg( \sum_{\gamma \in D_j^k} f^2(\gamma) w(\gamma) \Bigg)^{1/2} \Bigg( \sum_{\gamma \in S_j^k} (I^*\mu)^2(\gamma) w(\gamma) \Bigg)^{1/2}.$$

And now we apply (CS2) to $(x_j^k, y_j^k)$ to obtain

$$\sum_{\gamma \in S_j^k} (I^*\mu)^2(\gamma)\, w(\gamma) \le A^2 \cdot (I^*\mu)((x_j^k, y_j^k)), \text{ since}$$

$S_j^k = \{\beta \le (x_j^k, y_j^k)\}$. Next, $(I^*\mu)((x_j^k, y_j^k)) = \mu(S_j^k)$,

so, after plugging this into our estimate above, we arrive at

$$\mu(F_j^k) \le A \cdot 3^{-k} \Bigg( \sum_{\gamma \in D_j^k} f^2(\gamma) w(\gamma) \Bigg)^{1/2} \cdot \left(\mu(S_j^k)\right)^{1/2}.$$

We continue by summing over $j, k$,

$$\sum_{k,j} \mu(F_j^k) \cdot 3^{2k} \le A \cdot \sum_{k,j} 3^k \cdot \Bigg( \sum_{\gamma \in D_j^k} f^2(\gamma) w(\gamma) \Bigg)^{1/2} \left(\mu(S_j^k)\right)^{1/2} \overset{\text{Hölder}}{\le}$$

$$\le A \Bigg( \sum_{k,j} \sum_{\gamma \in D_j^k} f^2(\gamma) w(\gamma) \Bigg)^{1/2} \cdot \Bigg( \sum_{k,j} 3^{2k} \mu(S_j^k) \Bigg)^{1/2} \le \Big\{ \text{since } \sum_{k,j} \mathbb{1}_{D_j^k} \le 3 \Big\}$$

$$\leq C\cdot A \cdot \Big(\sum_{\gamma\in\Gamma}{}' f^2(\gamma)w(\gamma)\Big)^{1/2}\Big(\sum_{\gamma\in\Gamma}{}' \Big(\sum_{k,j}{}' 3^{2k}\mathbb{1}_{S_j^k}\Big)(\gamma)\mu(\gamma)\Big)^{1/2}$$

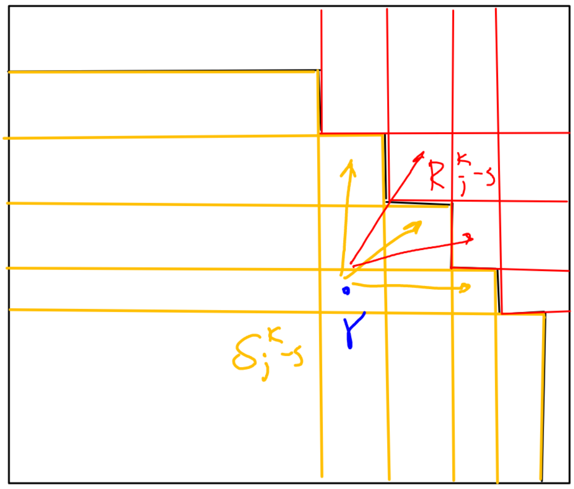
We are left with estimating the quantity in the right brackets. To do that we first notice that

$$\sum_j{}' \mathbb{1}_{S_j^k} \leq 3^{-k}\cdot \mathbb{1}_{\Omega_k} I_w f.$$

This should be fairly obvious after looking at the picture below – – if we have that $\gamma$ belongs to $N$ different rectangles $S_j^k$ for different $j$'s, then immediately $\gamma$ 'lies below' $N-1$ corresponding rectangles $R_j^k$ with same index set, and that, in turn, means that $(I_w f)(\gamma) \geq \frac{1}{3}\cdot 3^k\cdot N$, since $R_j^k$-rectangles are 'independent' in terms of $I_w$, that is

$$\sum_{\bigcup_{m=1}^{n} R^k_{j_m}\ni\gamma}{}' f(\gamma)w(\gamma) \approx \sum_{m=1}^{n}{}'\sum_{\gamma\in R^k_{j_m}}{}' f(\gamma)w(\gamma) = n\cdot 3^k \text{ for any collection}$$

$\{j_1,\dots,j_n\}$ of indices $j$.




Thus

$$\sum_{k,j}{}' 3^{2k}\mathbb{1}_{S_j^k}(\gamma) \leq C\sum_k{}' 3^k \mathbb{1}_{\Omega_k}(\gamma) I_w f(\gamma) \approx$$

$$\approx 3^{k_\gamma}\cdot I_w f(\gamma) \approx (I_w f)^2(\gamma),$$

where $k_\gamma = \max\{k:\gamma\in\Omega_k\} =$

$= \max\{k: I_w(\gamma) \geq 3^k\}.$

Combining the estimates we arrive at

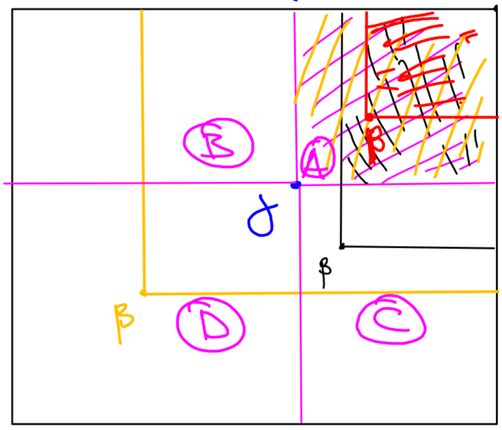
$$(\mathrm{I}) = \sum_{k,j}{}' 3^{2k}\cdot\mu(F_j^k) \leq C\cdot A\cdot \Big(\sum_{\gamma\in\Gamma}{}' f^2(\gamma)w(\gamma)\Big)^{1/2}\Big(\sum_{\gamma\in\Gamma}{}' (I_w f)^2(\gamma)\mu(\gamma)\Big)^{1/2}$$

And now comes the time to estimate the term $(\mathrm{II})$ – here we use our two remaining conditions, (CS1) and (CS3). We aim to show that $(\mathrm{II}) = \sum_{k,j}{}' 3^{2k}\mu(E_j^k) \leq C\cdot A\cdot \sum_{\gamma\in\Gamma}{}' f^2(\gamma)w(\gamma)$. Let us proceed.

Fix any point $\gamma \in \Gamma$. Then we claim that (CS1) and (CS3) imply the following estimate

$$\text{(CSM)} \quad \sum_{\beta\in\Gamma}{}' \Big( \sum_{\substack{\alpha\geq\gamma\\ \alpha\geq\beta}}{}' w(\alpha) \Big)^2 \mu(\beta) \leq C A^2 \cdot \sum_{\alpha\geq\gamma}{}' w(\alpha).$$

various possible positions of $\gamma$




Assuming $\gamma$ is given we see that $\beta \in \Gamma$ can fall into one of the four possible parts of $\Gamma$ (w.r.t. $\gamma$): either $\beta \geq \gamma$ (so part (A)), or $\beta \leq \gamma$ (part (D)), or $\beta$ is incomparable with $\gamma$ (parts (B) and (C)). Hence we write

$$\sum_{\beta\in\Gamma}{}' \Big( \sum_{\substack{\alpha\geq\gamma\\ \alpha\geq\beta}}{}' w(\alpha) \Big)^2 \mu(\beta) =$$

$$= \Big( \sum_{\beta\in (A)}{}' + \sum_{\beta\in (B)}{}' + \sum_{\beta\in (C)}{}' + \sum_{\beta\in (D)}{}' \Big) \Big( \sum_{\substack{\alpha\geq\gamma\\ \alpha\geq\beta}} w(\alpha) \Big)^2 \mu(\beta).$$

**(A)** By definition

$$\sum_{\beta\in (A)}{}' \Big( \sum_{\substack{\alpha\geq\gamma\\ \alpha\geq\beta}}{}' w(\alpha) \Big)^2 \mu(\beta) = \sum_{\beta\geq\gamma}{}' \Big( \sum_{\alpha\geq\beta}{}' w(\alpha) \Big)^2 \mu(\beta) =$$

$$= \sum_{\beta\geq\gamma}{}' (Iw(\beta))^2 \mu(\beta) \overset{\text{by (CS3)}}{\leq} A^2 \cdot (Iw)(\gamma).$$

**(D)** In this case, i.e. when $\beta \leq \gamma$, the sum $\sum_{\substack{\alpha\geq\beta\\ \alpha\geq\gamma}}{}' w(\alpha)$ is just $\sum_{\alpha\geq\gamma}{}' w(\alpha) = (Iw)(\gamma)$,

therefore

$$\sum_{\beta\in (D)} \Big( \sum_{\substack{\alpha\geq\beta\\ \alpha\geq\gamma}} w(\beta) \Big)^2 \mu(\beta) = \sum_{\beta\leq\gamma} (Iw)^2(\gamma) \mu(\beta) = (Iw)^2(\gamma) \cdot (I^*\mu)(\gamma) \leq$$

$$\overset{\text{by (CS1)}}{\leq} A^2 \cdot (Iw)(\gamma).$$

(B) & (C) Assume that $\beta \in$ (B) (the other case can be done in exactly the same way). Now this part of $\Gamma$ consists of such points $\beta$ that $\beta_1 \le \gamma_1$ and $\beta_2 \ge \gamma_2$, where $\beta_i, \gamma_i$ are corresponding coordinate vertices in $\Gamma_1, \Gamma_2$ respectively. In particular, if $\tau \in$ (B) and $\tau_2 = \beta_2$, then

$$\sum_{\substack{\alpha \ge \beta \\ \alpha \ge \gamma}}' w(\alpha) = \sum_{\substack{\alpha_1 \ge \gamma_1 \\ \alpha_2 \ge \beta_2}}' w(\alpha) = \sum_{\substack{\alpha_1 \ge \gamma_1 \\ \alpha_2 \ge \tau_2}}' w(\alpha) = \sum_{\substack{\alpha \ge \tau \\ \alpha \ge \gamma}}' w(\alpha).$$

It follows that

$$\sum_{\beta \in (B)}' \Big(\sum_{\substack{\alpha \ge \beta \\ \alpha \ge \gamma}}' w(\alpha)\Big)^2 \mu(\beta) = \sum_{\beta \in (B)}' \Big(\sum_{\substack{\alpha_1 \ge \gamma_1 \\ \alpha_2 \ge \beta_2}}' w(\alpha)\Big)^2 \mu(\beta) = \sum_{\substack{\beta_1 \le \gamma_1 \\ \beta_2 \ge \gamma_2}}' \Big(\sum_{\substack{\alpha_1 \ge \gamma_1 \\ \alpha_2 \ge \beta_2}}' w(\alpha)\Big)^2 \mu(\beta) =$$

$$= \sum_{\beta_2 \ge \gamma_2}' \left( \Big(\sum_{\substack{\alpha_1 \ge \gamma_1 \\ \alpha_2 \ge \beta_2}}' w(\alpha)\Big)^2 \cdot \Big(\sum_{\beta_1 \le \gamma_1}' \mu(\beta)\Big) \right) = \sum_{\beta_2 \ge \gamma_2}' \Big(\sum_{\alpha_2 \ge \beta_2}' w_2(\alpha_2)\Big)^2 \cdot \mu_2(\beta_2),$$

where $w_2(\alpha_2) = \sum_{\alpha_1 \ge \gamma_1}' w(\alpha_1, \alpha_2)$ and $\mu_2(\beta_2) = \sum_{\beta_1 \le \gamma_1}' \mu(\beta_1, \beta_2) \cdot \mathbb{1}_{\mathcal{P}(\gamma_2)}$

So now we are back to the 1d-situation, where the iff condition is just the 1d-version of (CS1). Writing the boundedness of the embedding

$$\sum_{\beta_2 \in \Gamma_2}' \big(I_{w_2} f\big)^2(\beta_2)\, \mu_2(\beta_2) \le C \cdot \sum_{\beta_2 \in \Gamma_2}' f^2(\beta_2)\, w_2(\beta_2)$$

and substituting $f = \mathbb{1}_{\mathcal{P}(\gamma_2)}$, we get

$$\sum_{\beta_2 \ge \gamma_2}' \Big(\sum_{\alpha_2 \ge \gamma_2}' w_2(\alpha_2)\Big)^2 \mu_2(\beta_2) \le C \cdot \sum_{\beta_2 \ge \gamma_2}' w_2(\beta_2),$$

which is just another way to write (CS1) for 1-dimensional case with a particular choice of the weight-measure pair $(\mu_2, w_2)$.

But this condition is

$$\sup_{\beta_2\in\Gamma_2} (I w_2)(\beta_2)\,(I^*\mu_2)(\beta_2) \le A^2, \text{ which is}$$

$$A^2 \geq \sup_{\beta_2\in\Gamma_2} \Big(\sum_{\alpha_2\geq\beta_2} w_2(\alpha_2)\Big)\Big(\sum_{\alpha_2\leq\beta_2} \mu_2(\alpha_2)\Big) =$$

$$= \sup_{\beta_2\in\Gamma_2}\Big(\sum_{\alpha_2\geq\beta_2}\sum_{\alpha_1\geq\gamma_1} w(\alpha_1,\alpha_2)\Big)\Big(\sum_{\alpha_2\leq\beta_2}\sum_{\alpha_1\leq\gamma_1}\mu(\alpha_1,\alpha_2)\Big) =$$

$$= \sup_{\beta_2\in\Gamma_2} (Iw)(\gamma_1,\beta_2)\cdot(I^*\mu)(\gamma_1,\beta_2),$$

and this inequality is of course true being a direct corollary of (CS1). Hence

$$\sum_{\beta\in B}\Big(\sum_{\substack{\alpha\geq\gamma\\ \alpha\geq\beta}} w(\alpha)\Big)^2\mu(\beta) \le C\cdot (Iw)(\gamma),$$

and the same argument works for $E$ as well, and (CSM) is proven.

---

We can proceed with (II). We recall that $I_w f = 3^{k+1}$ on the boundary $\partial\Omega_{k+1}$ of $\Omega_k$, and that the 'step' rectangles $R_j^k$ have their lower left corners on $\Omega_{k+1}$, thus

$$\sum_{k,j} 3^{2k}\mu(E_j^k) \sim \sum_{k,j}\mu(E_j^k)\cdot\Big(\sum_{\gamma\in R_j^k} f(\gamma)w(\gamma)\Big)^2 =$$

$$= \sum_{k,j}\mu(E_j^k)\cdot w^2(R_j^k)\cdot\Big(\frac{1}{w^2(R_j^k)}\sum_{\gamma\in R_j^k} f(\gamma)w(\gamma)\Big)^2,$$

where $w(R_j^k) = (Iw)(x_{j+1}^k, y_j^k) = \sum_{\gamma\in R_j^k} w(\gamma)$, and the last term above is a normalized integral of $f$ over $R_j^k$.

---

Up until now we were using a certain partition of $\Gamma$ via the sets $\Omega_k$ – the level sets of $I_w f$. But now we introduce another way to look at points of $\Gamma$, which comes from the distribution of values of 'normalized' Hardy operator $\frac{I_w}{I_w}$, which appears in the right-hand side of the computation above. This new distribution actually makes the rest of arguments.

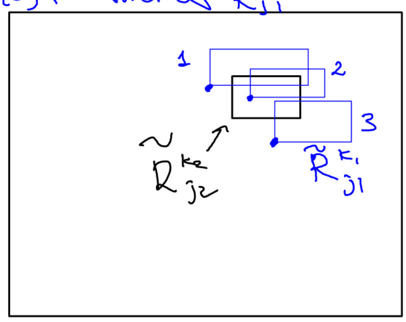
---

So, for an integer $\ell$ define by $\Delta_\ell$ the set of pairs $(k,j)$ such that

$$\frac{1}{\sum_{j\in R_j^k} w(j)} \sum_{j\in R_j^k} f(j)\, w(j) < 2^{\ell}.$$

Now for a given $\ell$ consider a collection $\{U_i^\ell\}$ of maximal rectangles from $\{R_j^k\}_{(k,\ell)\in\Delta_\ell}$. Now these rectangles themselves are not disjoint, however we can modify them to be such. Namely, for every $U_i^\ell$ (which is just some $R_j^k$ with $(k,j)\in\Delta_\ell$) consider $\tilde{U}_i^\ell$ which is the respective $\tilde{R}_j^k$ (its lower left corner). These ones are, in fact, disjoint (for a given $\ell$). Indeed, let $(k_1,j_1), (k_2,j_2)\in\Delta_\ell$, and, say, $k_1\le k_2$. Assume now that $\tilde{R}_{j_1}^{k_1}\cap\tilde{R}_{j_2}^{k_2}$. That means that the lower left corner $(x_{j_1+1}^{k_1}, y_{j_1}^{k_1})$ of $\tilde{R}_{j_1}^{k_1}$ must either inside $\tilde{R}_{j_2}$, or below its lower side, or left of its left side. Well, this can not happen, since if $k_1=k_2$, then $\tilde{R}_{j_1}^{k_1}$ obviously is disjoint from $\tilde{R}_{j_2}^{k_2}$. Otherwise (positions 1 and 3) we have that $(I_w f)(x_{j_1+1}^{k_1}, y_{j_1}^{k_1}) = 3^{k_1+1} < 3^{k_2}$, hence $(x_{j_1+1}^{k_1}, y_{j_1}^{k_1}) \notin \Omega_{k_2}$. The only possible option to put $(x_{j_1+1}^{k}, y_{j_1}^{k_1})$ is to somehow make it inside $\tilde{R}_{j_2}^{k_2}\setminus\Omega_{k_2}$, but this would not work either — in this case $R_{j_1}^{k_1}\subset R_{j_2}^{k_2}$, and we have a contradiction with the maximality condition for $\Delta_\ell$.

possible location of the lower left corner of $\tilde{R}_{j_1}^{k_1}$



The second ingredient is the following inequality

$$2^{e-3} < \frac{1}{w(U_i^e)} \sum'_{\substack{\gamma\in \tilde U_i^e\\ \gamma:\, f(\gamma)>2^{e-3}}} f(\gamma)\, w(\gamma), \qquad \forall e, i. \quad (*)$$

This happens because, $\tilde U_i^e$ is just $\tilde R_j^k$ for some $(k,j)\in\Delta_e$

$$\sum'_{\gamma\in\tilde U_i^e} f(\gamma)w(\gamma) = \sum'_{\gamma\in\tilde R_j^k} f(\gamma)w(\gamma) \geq 3^{k+1}-3^k-3^k = 3^k =$$

$$= \frac{1}{3} \sum'_{\gamma\in R_j^k} f(\gamma)\, w(\gamma) > 2^{e-2} w(R_j^k).$$ The restriction $\{\gamma: f(\gamma)>2^{e-3}\}$ is obvious as well.

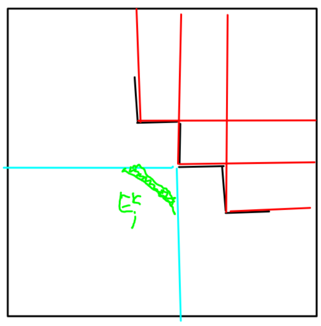
---

Now, for a fixed $e$ we have

$$\sum'_{(k,j)\in\Delta_e} \mu(E_j^k)\, w^2(R_j^k) \le \sum_i \sum'_{\substack{(k,j):\\ R_j^k\subset U_i^e}} \mu(E_j^k)\, w^2(R_j^k)$$

Next, all $E_j^k$ are disjoint, and if $R_j^k\subset U_i^e$, then

$$\left(I_w(\mathbb{1}_{U_i^e})\right)(\gamma) = \sum'_{\substack{\beta\geq\gamma\\ \beta\in U_i^e}} w(\beta) \geq \sum'_{\beta\in R_j^k} w(\beta), \quad \gamma\in E_j^k,$$




since in this case $\gamma$ lies below and to the left of the lower left corner $(x_{j+1}^k, y_j^k)$ of $R_j^k$ (in other words, $\gamma\le(x_{j+1}^k, y_j^k)$). Hence

$$\sum'_{(k,j):\, R_j^k\subset U_i^e} \mu(E_j^k)\, w^2(R_j^k) \le \sum'_{\substack{\gamma\in E_j^k:\\ (k,j):\, R_j^k\subset U_i^e}} \mu(\beta)\cdot\left(I_w(\mathbb{1}_{U_i^e})(\beta)\right)^2 \le$$

$$\leq \sum_{\beta\in\Gamma}{}' \mu(\beta)\cdot\left(I_w\left(\mathbb{1}_{U_i^e}\right)\right)^2(\beta) = \left\{\text{let } \gamma_i^e \text{ be the lower left corner of } U_i^e\right\} =$$

$$= \sum_{\beta\in\Gamma}{}'\Big(\sum_{\substack{\alpha\geq\beta\\ \alpha\geq\gamma_i^e}}{}' w(\alpha)\Big)^2\mu(\beta) \overset{(CSM)}{\leq} C\cdot(I\,w)(\gamma_i^e) = C\cdot w(U_i^e).$$

Thus
$$\sum_i \sum_{\substack{(k,j):\\ R_j^k\subset U_i^e}}{}' \mu(E_j^k)\,w^2(R_j^k) \leq C\cdot\sum_i{}' w(U_i^e) \leq \{\text{by } (*)\}$$

$$\leq C\cdot\sum_i{}'\Big(2^{-e}\cdot\sum_{\substack{\gamma\in\tilde U_i^e\\ f(\gamma)>2^{e-3}}}{}' f(\gamma)w(\gamma)\Big) \leq \{\tilde U_i^e \text{ are disjoint}\} \leq$$

$$\leq C\cdot 2^{-e}\cdot\sum_{\gamma:\, f(\gamma)>2^{e-3}} f(\gamma)w(\gamma).$$

Now, finally,
$$\sum_{k,j}{}'\mu(E_j^k)\cdot 3^{2k} \leq C\cdot\sum_e{}' 2^{2e}\sum_{(k,j)\in\Delta_e}{}' \mu(E_j^k)\,w^2(R_j^k) \leq$$

$$\leq C\cdot\sum_e{}' 2^e\cdot\sum_{\gamma:\, f(\gamma)>2^{e-3}}{}' f(\gamma)w(\gamma) \leq C\cdot\sum_{\gamma\in\Gamma}{}' f^2(\gamma)w(\gamma).$$

---

As a result we obtained the final estimate
$$\sum_{\gamma\in\Gamma}{}'(I_wf)^2(\gamma)\mu(\gamma) \leq C\cdot\Big(\sum_{k,j}{}' 3^{2k}\mu(E_j^k) + \sum_{k,j}{}' 3^{2k}\mu(F_j^k)\Big) \leq$$

$$\leq C\cdot\Big(\Big(\sum_{\gamma\in\Gamma}{}' f^2(\gamma)w(\gamma)\Big)^{1/2}\Big(\sum_{\gamma\in\Gamma}{}'(I_wf)^2(\gamma)\mu(\gamma)\Big)^{1/2} +$$

$$+ \sum_{\gamma\in\Gamma}{}' f^2(\gamma)w(\gamma)\Big)$$ which immediately implies our desired embedding

$$\boxed{\sum_{\gamma\in\Gamma}(I_wf)^2(\gamma)\mu(\gamma) \leq C\cdot\sum_{\gamma\in\Gamma}{}' f^2(\gamma)w(\gamma) \text{ with } C\sim A^2.}$$

## Problems

- Could we do it on 3d-lattice? There are several dimension locked arguments here — step-ladder construction, independence of $I_\omega$ over different $R_j^k$-s — we don't see how to increase the dimension.
- As a possible way around one can try to think of a different arguments that are, so to say, less dependent, like maybe a capacitory-type estimates. No luck yet finding one though.
- What about a proper bi-tree? There one also would need a non-single box type argument, while somehow dealing with a more complicated bi-tree structure.

A hypothesis: we employ two different partitions of $T$. The first one is through level sets of $I_\omega f$ (that is somehow like the polar change of variables, which is often used to prove the 'classical' Hardy inequality), and also the 'symmetric' level sets of $I_\mu^* \varphi$ — a weighted adjoint Hardy operator.

A lot of things to do still.