Weekly seminar on the stack of formal groups
Preliminary program

Organisers: Andrey Lavrenov, Pavel Sechin*

Starting June 2020 via Zoom (in Russian)

Formal group laws appear in topology through 'orientations' of generalized cohomology theories. Since the introduction of Morel-Voevodsky motivic homotopy theory, the same can be said about algebraic geometry. One way to express this is as a mathematical statement is as follows: there is a canonical isomorphism between Lazard ring and the coefficient ring of (algebraic or complex) cobordism. Every morphism from cobordism to another cohomology theory A, thus, induces a formal group law on the coefficient ring of A. In fact, from certain perspective[1] there is a 1-to-1 correspondence between formal group laws and cohomology theories.

However, it turns out that this correspondence has more functoriality than just a map of sets. The most well-investigated part of this more general relation concerns strict isomorphisms between formal group laws which are nothing else than change of the 'coordinate' of the formal group law. On the level of cohomology theories strict isomorphisms correspond to stable multiplicative operations. This new functoriality studies not just sets of formal group laws over a particular ring (which are (co)represented by Lazard ring $\mathbb{L}$) but groupoids of formal group laws with strict isomorphisms between them over a particular ring. Presheaves of groupoids on the category of affine schemes sometimes can be represented by what is called a Hopf algebroid, and the main goal of this seminar will be the study of the Hopf algebroid $(\mathbb{L}, \mathbb{L}B)$ corepresenting formal groups.

One can study this Hopf algebroid purely algebraically in the same way as one can study affine algebraic groups as Hopf algebras. However, there is a geometric point view that goes through associating to every Hopf algebroid an algebraic[2] stack. As usual, this change of point of view does not necessarily ease the burden of proving claims about Hopf algebroids and associated structures, but one may argue that the gain is in certain clarity obtained from this perspective. Just as one example of what is meant here, is the following result of Hovey-Strickland [HS, Theorem C]: the categories of $E(n)_*E(n)$-comodules, $(v_n^{-1}BP)_*(v_n^{-1}BP)$-comodules are all equivalent as abelian categories for a particular prime p and n (note that $E(n) \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots v_n^{\pm 1}]$ depends as an algebra over Lazard ring on the choice of v_n and also on the images of v_i, $i > n$). The geometric interpretation of this statement is explained in [Na07]: these categories are all canonically equivalent to the category of quasi-coherent sheaves on a particular 'open' substack of the stack of formal groups. In fact, this geometric interpretation allows one to prove that these categories are equivalent not just as abelian categories, but as tensor abelian categories. Having said that, it might be less surprising that the second goal of the seminar will be to understand known results about the Hopf algebroid of formal group laws and related structures (e.g. comodules) through this geometric picture (in particular, we

*pavel.sechin@gmail.com; Please write me an e-mail if you want to participate.

[1] namely, of free theories or, which is the same, of theories of rational type

[2] in certain sense; though, in general, not in the classical sense of Artin

hope to cover the theorem of Hovey-Strickland as proved by Naumann). Since stacks are not yet a standard part of the algebraic geometry education, no knowledge of the stacks will be assumed.

Finally, if we succeed, there will be a series of talks that will use and illustrate some of the obtained results on the stack of formal group law in the study of algebraic cobordism of Levine-Morel.

1 Syllabus

Stacks

1. groupoids, Hopf algebroids, pre-stacks;
2. (algebraic) stack associated to a Hopf algebroid;
3. the stack of formal groups (also, globally by explaining what a formal group is);
4. quasi-coherent sheaves on stacks, descent properties;
5. properties of morphisms between stacks;
6. tangent space to a stack.

FGL's

1. FGL's: examples, logarithm, endomorphisms over fields of positive characteristic;
2. ? Lazard's theorem on the structure of Lazard ring;
3. Lazard's theorem on FGLs over fields of positive characteristic;
4. p-typical formal group laws and p-typization;
5. Hopf algebroids MU_*MU and BP_*BP;
6. Landweber invariant prime ideals;
7. primary decomposition of comodules and their homological properties;
8. Lubin-Tate deformation theory.

Applications (very tentative)

1. algebraic cobordism of Pfister quadrics;
2. categories of Morava motives and relations between them;
3. ???

2 Plan

The talks about stacks and formal group laws are more or less independent until Talk 8, and their order can be changed without much pain.

Talk 1. Introduction to formal group laws.

Define formal group laws (FGLs) and morphisms between them, prove the existence of Lazard ring $\mathbb{L}$.
Study FGLs over $\mathbb{Q}$-algebras, prove the existence of the logarithm.
Study FGLs over $\mathbb{F}_p$-algebras, define the height and prove that it is an invariant under isomorphisms.
Provide examples of FGLs of all heights (aka Lubin-Tate formal group laws?).
Compute endomorphism algebra of an FGL of height n over $\overline{\mathbb{F}}_p$.

Main references: [Rav03, Appendix A2], [Haz78, Chapter 1].

Talk 2. Introduction to stacks I.

Recall the definition of groupoids, consider abstract groups as examples.
Show that given a morphism $f : X \to Y$ there is a natural structure of a groupoid on $X, X \times_Y X$.
Define the 2-category of groupoids, explain the notion of (co)limits in it. Illustrate it with examples of fiber products of morphisms between classifying groupoids, e.g. $G/H \to BH \leftarrow *$ or $BH \to BG \leftarrow *$.
Give and explain the definition of a Hopf algebroid. Explain relations between Hopf algebras and Hopf algebroids.
If time permits, consider a Hopf algebroid corresponding to an action of an algebraic group on an affine variety.
Define pre-stacks as presheaves of groupoids. Explain the Grothendieck construction that relates presheaves of groupoids with categories fibred in groupoids.

Main references: [Ol16, Chapter 3], [Rav03, Appendix A1].

Talk 3. Structure of the Lazard ring and its generators.

Prove that the Lazard ring is (non-canonically) isomorphic as a graded ring to a polynomial ring $\mathbb{Z}[t_1, t_2 \ldots]$ with variables t_i with $\deg t_i = -i$. In particular, you should construct 'Hurewicz morphism' $H : \mathbb{L} \to \mathbb{Z}[b_1, b_2, \ldots]$ and prove its injectivity.
For a natural number n define the Milnor genus or the Milnor characteristic number $s_n : \mathbb{L}^{-n} \to \mathbb{Z}$ as the composition of H with the projection to the coefficient of b_n. Show that the image of s_n is divisible by p if $n = p^k - 1$ for some k, or is the whole group $\mathbb{Z}$ otherwise. Prove that a set of elements $\{x_i | i = 1, 2, \ldots, x_i \in \mathbb{L}^{-i}\}$ is a generating set of $\mathbb{L}$ as algebra if and only if $s_i(x_i)$ is the generator of the group $s_i(\mathbb{L}^{-i})$.
If time permits, you might explain the relation between Hurewicz morphism and characteristic numbers of smooth projective varieties (or manifolds) as well as construct algebrogeometric generators of $\mathbb{L}_{(p)}$ for each prime p.

Main references: [Rav03, Theorem A2.1.10], [Lu10, Lectures 2,3].

Talk 4. Introduction to stacks II.

Explain the descent condition for a presheaf of groupoids (w.r.to some Grothendieck topology).
Explain that there is a process of 'sheafification' (perhaps, without all the details).
Define stacks and provide examples, mainly, quotient stacks.
Define algebraic stacks and explain how one can define étale-local properties of morphisms between stacks.

Main reference: [Ol16, Chapters 4,8].

Talk 5. Formal group laws over a field of positive characteristic.

Prove the theorem of Lazard that a formal group law over an algebraically closed field of characteristic p is uniquely defined up to a strict isomorphism by its height.

Explain how one can extend this classification to perfect fields using Galois theory.

If time permits, provide examples of non-isomorphic formal group laws over $\mathbb{F}_{(p)}$ of certain height.

Main references: [Rav03, Theorem A2.2.11][3] *[Lu10, Lecture 14].*

Talk 6. The stack of formal groups $\mathcal{M}_{fg}$.

We have already defined the stack of formal groups using a general construction which associates a stack to a Hopf algebroid. However, there is a much more concrete description for morphisms from affine schemes to $\mathcal{M}_{fg}$, i.e. there is a definition of a formal group over a commutative rings which satisfies descent.

To start the geometric description of $\mathcal{M}_{fg}$ explain the correspondence between closed reduced substacks of $\mathcal{M}_{fg}$ and Landweber invariant ideals of $\mathbb{L}$. What are the the geometric 'points' of $\mathcal{M}_{fg}$?

Main references: [Lu10, Lecture 11], [Na07, Section 6].

Talk 7. p-typical formal group laws.

Define p-typical formal group laws and prove that over $\mathbb{Z}_{(p)}$-algebra there is a canonical isomorphism Existence of p-typization, example of Artin-Hasse exponent, Araki and Hazewinkel generators of BP and p-summation in BP.

Define Hopf algebroids (BP, BP_*BP) and $(\mathbb{L}, \mathbb{L}B)$ in terms of formal group laws and prove structural results about them.

If time permits, explain equivalences of Hopf algebroids on the example of (BP, BP_*BP) and $(\mathbb{L}_{(p)}, \mathbb{L}_{(p)}B)$.

Main references: [Rav03, Section A2.1], [Haz78, Chapter III].

Talk 8. Closed reduced substacks of $\mathcal{M}_{fg}$are Cartier divisors.

The goal of this talk is two-fold. First, you have to describe all closed reduced substacks of $\mathcal{M}_{fg}$. Second, you have to show that they are in certain sense Cartier divisors (explaining what it means).

Main references: TBA.

Talk 9. Quasi-coherent sheaves on stacks.

Describe the notion of a quasi-coherent sheaf on an (algebraic) stack and explain descent properties for the category $QCoh$.

There are two main goals of this talk.

First, relate the categories $QCoh(X)$ with $(A, B) - Comod$ where (A, B) is a flat Hopf algebroid and X is the associated stack.

Second, explain the primary decomposition of $(\mathbb{L}, \mathbb{L}B)$-comodules. Perhaps, it's better if you can prove a more general primary decomposition result for stacks and apply it here.

Main references: [Ol16, Chapter 9], [Na07, Section 3].

Talk 10. Tangent space to a stack. Deformation theory. Lubin-Tate theory.

TBA.

Talks 12-. . . Applications: TBA.

[3] Ravenel's exposition uses the theory of p-typical formal group laws, it's better to exchange Talks 5, 7,if one wants to follow this source.

References

[Haz78] M. Hazewinkel *Formal groups and applications*, Vol. 78. Elsevier, 1978.

[HS] M. Hovey, N. Strickland *Comodules and Landweber exact homology theories*, Advances in Mathematics 192.2 (2005): 427-456.

[La73a] Landweber P. S. *Associated prime ideals and Hopf algebras*, Journal of pure and applied algebra 3.1 (1973): 43-58.

[La73b] Landweber P. S., *Annihilator ideals and primitive elements in complex bordism*, Illinois Journal of Mathematics 17(2) (1973): 273-284.

[La76] Landweber P. S. *Homological properties of comodules over MU*(MU) and BP*(BP)*, American Journal of Mathematics (1976): 591-610.

[Na07] Naumann N. *The stack of formal groups in stable homotopy theory*, Advances in Mathematics 215.2 (2007): 569-600.

[Lu10] J. Lurie *Chromatic homotopy theory*, Lecture notes online (2010) Link to Lecture 1(change address accordingly).

[Ol16] M. Olsson *Algebraic spaces and stacks*, Vol. 62. American Mathematical Soc., 2016.

[Rav03] D. C. Ravenel *Complex cobordism and stable homotopy groups of spheres*, American Mathematical Soc., 2003.

[Str] N. Strickland *Formal groups*, unfinished notes, avaialble here ???