The Beurling-Wintner problem and analytic number theory

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Abstract: This talk concerns a long-standing problem on completeness of function systems generated by odd periodic extensions of functions in $L^2(0,1)$. This problem, raised by Beurling and Wintner in the 1940s, is closely related to the Riemann Hypothesis. We completely solve the rational version of step functions(that is, for those functions with rational jump discontinuities) by approaches from analytic number theory, and present several deep applications including a complete solution to the rational version of Kolzov completeness problem. This is a joint work with H.Dan.

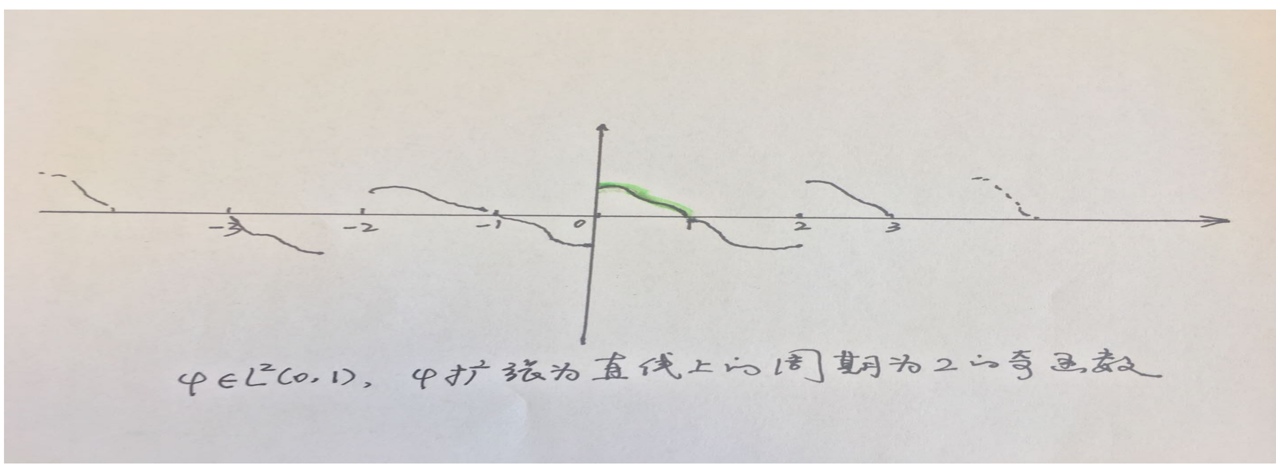
Preview

- The Beurling-Wintner problem
- The Kolzov completeness problem
- The Beurling – Wintner transformation
- Dirichlet characters and Dirichlet L-functions
- The cyclicity of translations of Dirichlet series
- The B-W problem for step functions
- Jumps and discontinuities for step functions
- The B-W problem for characteristic functions
- The B-W problem for real step functions

1. The Beurling-Wintner problem

The standard Lebesgue space $L^2(0,1)$: Let $\{\varphi_1, \varphi_2, \cdots,\}$ be a sequence in $L^2(0,1)$, the sequence is said to be complete if the linear span of $\{\varphi_n(x)\}_{n\geq 1}$ is dense in $L^2(0,1)$.

For each φ in $L^2(0,1)$, φ is considered as a function on the whole real line by extending φ to an odd periodic function of period 2.



The classical Beurling-Wintner problem raised by Beurling (1945) and independently by Wintner (1944-1945) asked for which $\varphi \in L^2(0,1)$, the dilation system $\{\varphi(kx) : k = 1, 2, \cdots\}$ is complete in $L^2(0,1)$, where φ is identified with its extension to an odd 2-periodic function on $\mathbb{R}$. This difficult problem is nowadays called as the Beurling-Wintner Problem (by Nikolski).

The B-W problem: for which function $\varphi \in L^2(0,1)$, the dilation system $\{\varphi(x), \varphi(2x), \cdots\}$ is a complete sequence in $L^2(0,1)$.

The prototype is $\varphi(x) = \sqrt{2}\sin \pi x$, sine $\{\varphi(kx) = \sqrt{2}\sin(k\pi x) : k \in \mathbb{N}\}$ is a canonical orthonormal basis of $L^2(0,1)$.

The dilation system $\{\varphi(nx)\}_{n\geq 1}$ is closely linked to various problems in mathematics. For examples,

- In analytic number theory, let $\rho(x) = x - [x]$, then

$$1,\ \rho(x),\ \rho(2x),\ \cdots,\ \rho(nx),\ \cdots$$

  is complete in $L^2(0,1)$. This sequence results from an appropriate form of an analytic version of the sieve of Eratosthenes.

- Considering a nonlinear equation (for the one-dimensional p-Laplacian):

$$\frac{d}{dx}(|f'|^{p-2}f') + \lambda|f|^{p-2}f = 0$$

where $1 < p < \infty$, It has a remarkable solution, and all solutions can be expressed in terms of dilations of this solution.

- The third is The Riemann hypothesis. The Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ has convergence region $\mathbf{Re}(s) > 1$.

  Set $\eta(s) = \sum_{n=1}^{\infty}(-1)^n n^{-s}$, then $\eta(s)$ is an entire function and

$$\eta(s) = 2(2^{-s} - 2^{-1})\zeta(s).$$

  Therefore the Riemann zeta function $\zeta(s)$ can continue analytically in the complex plane except for a unique simple pole $z = 1$.

  The Riemann hypothesis, considered one of the greatest unsolved problems in mathematics, asserts that any non-trivial zero point s lies on $\mathbf{Re}(s) = 1/2$.

We first mention Nikolski's paper "In a shadow of the RH,..." in Ann. Inst. Fourier, 2012. This is a significant paper on the dilation completeness problem. This paper illustrated the relation to the Riemann hypothesis (RH).

Indeed, the following statement about dilations of a nonperiodic function is equivalent to RH. Set $\varphi(x) = \frac{1}{x} - [\frac{1}{x}],\ x > 0,$

The Nyman – Baez-Duarte criterion: The characteristic function $\chi_{[0,1]}$ is in the closure in $L^2(0, \infty)$ of the system

$$\{\varphi(x), \varphi(2x), \varphi(3x), \cdots\}.$$

In a communication with Nikolski, Nikolski asked me whether ones can find a function $\varphi \in L^2[0, 1]$ such that completeness of the dilation system$\{\varphi(x), \varphi(2x), \varphi(3x), \cdots\}$ is equivalent to RH.

All these problems are all associated with completeness of the dilation system $\{\varphi(nx)\}_{n\geq 1}$ in $L^2(0,1)$. This difficult problem is nowadays called as the B-W Problem.

The study of the B-W problem can be dated back to P.Chebyshev (1859), A.Markov (1898), next A.Wintner (1944), A.Beurling (1945), N.Romanov (1944-1946), D.Bourgin (1946), N.Akhiezer (1940-1947), V.Ya.Kozlov (1948-1950), $\cdots$, H.Helson, N.Nikolski, K.seip, B.Mityagin, H.Hedenmalm, P.Lindqvist, E. Saksman, S.Noor, ... (and possibly others).

Two significant papers and two talks:

N. Nikolski, In a shadow of the RH: Cyclic vectors of Hardy spaces on the Hilbert multidisc, Ann. Inst. Fourier, Grenoble, 62(2012), 1601-1625.

N. Nikolski, The current state of the dilation completeness problem, Chebyshev Lab, SPb, 2018. （Talk）

H. Hedenmalm, P. Lindqvist and K. Seip, A Hilbert space of Dirichlet series and systems of dilated functions in $L^2(0;1)$, Duke Math. J. 86(1987), 1-37.

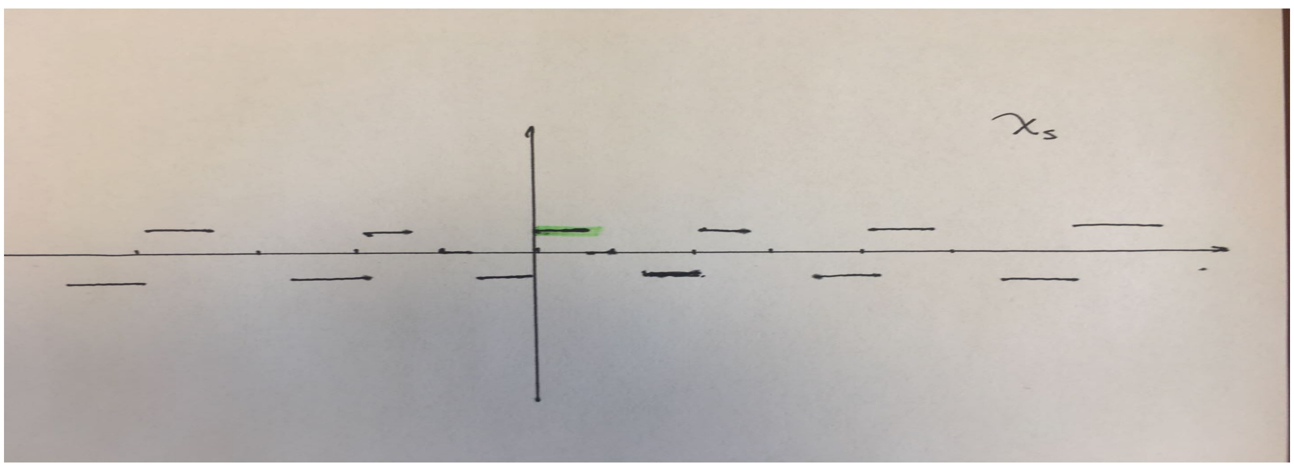
K. Seip, Hardy spaces of Dirichlet Series and the Riemann Zeta Function, Valencia, October 20, 2017.（Talk）

For a general φ, characterizing completeness of the dilation system $\{\varphi(nx)\}_{n\geq 1}$ is almost impossible. Therefore ones turn to some special functions.

2. The Kozlov completeness problem

V. Kozlov (1914-2007), Published a series of brilliant papers on the B-W problem in Doklady Akad. Nauk SSSR in 1948-1951.

Given a number $0 < s \leq 1$, and let χ_s be the characteristic function of $(0, s)$, and $\mathcal{D}_s = \{\chi_s(kx) : k = 1, 2, \cdots\}$ be the dilation system defined by the characteristic function χ_s.



Kozlov claimed some astonishing results in ”Doklady” papers without proof. He claimed that

1) the dilated system $\mathcal{D}_s$ is complete for $s = 1, \frac{1}{2}, \frac{2}{3}$; and is incomplete for s in a neighborhood of $\frac{1}{3}$;

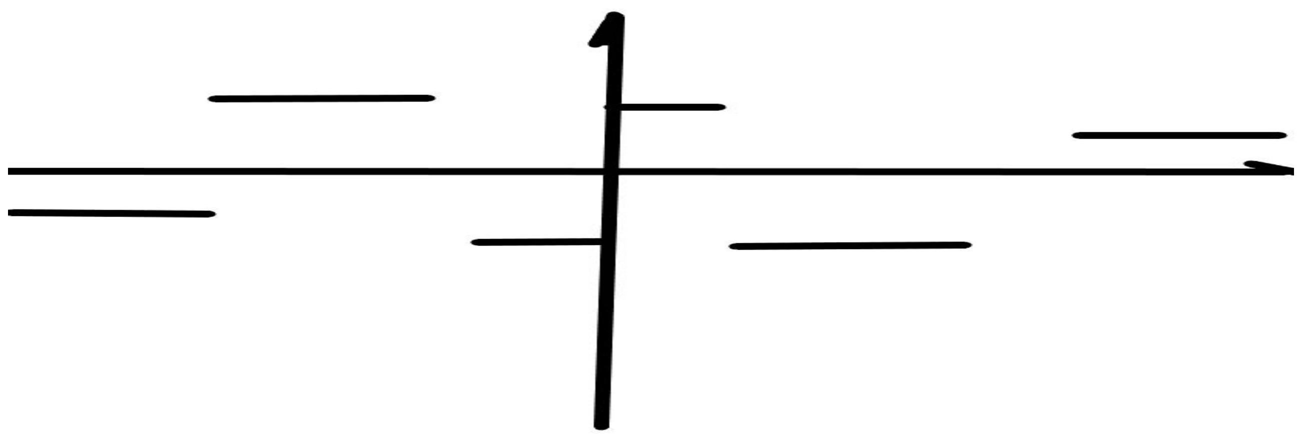
2) $\mathcal{D}_s$ is incomplete for $s = \frac{q}{p}$, where p is an odd prime and q is odd so that $\tan^2 \frac{q\pi}{2p} < \frac{1}{p}$.

The Kozlov problem is to decide for which s, the dilated system $\mathcal{D}_s$ is complete in $L^2(0, 1)$.（1948-1950）

Kozlov's proofs are not published until now. For quite a long time, only the case $r = 1$ was reproved in 1965 by Ahiezer with a long proof.

Until recently, Kozlov's claims were reproved by Nikolski. The proofs were exhibited in his talk in 2018. Indeed, for claim 2), he proved a stronger version, where the condition $\tan^2 \frac{q\pi}{2p} < \frac{1}{p}$ was replaced by $\sin^2 \frac{q\pi}{2p} < \frac{1}{p+1}$. Moreover, he also proved when $s = 1/4$, $\mathcal{D}_s$ is incomplete.

For step functions with rational jump discontinuities, we completely solve the B-W problem by approaches from analytic number theory, which results in a complete solution to the rational version of the Kolzov problem. Moreover, since all such step functions form a densely linear subspace of L^2, this enables us to draw conclusions for completeness of dilation systems of "almost all" functions by approximation strategy. Before continuing we need to do a lot of preparatory work.



3. The Beurling – Wintner transformation

Suppose that $\varphi \in L^2$ has its Fourier-sine expansion

$$\varphi(x) = \sum_{n=1}^{\infty} a_n \sqrt{2} \sin n\pi x, \quad 0 < x < 1.$$

Beurling and Wintner's idea to associate φ to the Dirichlet series

$$\mathcal{D}\varphi(s) := \sum_{n=1}^{\infty} a_n n^{-s}.$$

This map, called the Beurling – Wintner transformation (by K.Seip) satisfies

$$\mathcal{D}(\varphi(nx)) = n^{-s} \cdot \mathcal{D}\varphi. \qquad (\spadesuit)$$

That is, a dilations of φ is mapped to multiplication by Dirichlet monomial. A Dirichlet polynomial is $\sum_{n=1}^{N} c_n n^{-s}$.

The Hardy space $\mathcal{H}^2$of Dirichlet series consist of all Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad \text{and} \quad \|f\|^2 = \sum_n |a_n|^2 < \infty.$$

Since by Cauchy – Schwarz inequality, $|f(s)|^2 \leq \|f\|^2 \sum_n n^{-2\sigma}$, where $s = \sigma + it$, a complex variable, any $f \in \mathcal{H}^2$ is analytic in $\mathbb{C}_{1/2}$. Where for a real number σ,

$$\mathbb{C}_\sigma = \{z \in \mathbb{C} : \mathrm{Re}(z) > \sigma\}.$$

For each Dirichlet series D, there exists a real number $\sigma_a(D)$ (called the abscissa of absolute convergence of D) such that if $\sigma > \sigma_a(D)$, the series D converges absolutely, but not if $\sigma < \sigma_a(D)$. Therefore for each $D \in \mathcal{H}^2$, we have $\sigma_a(D) \leq 1/2$.

Notice that the Beurling – Wintner transformation $\mathcal{D}$ is a unitary transformation from L^2 onto $\mathcal{H}^2$, and $\mathcal{D}(\varphi(nx)) = n^{-s} \cdot \mathcal{D}\varphi$.

A Dirichlet series $D \in \mathcal{H}^2$ is called a cyclic vector if the multiplier invariant subspace generated by D is the whole space $\mathcal{H}^2$, that is,

$$\overline{\text{span}}\{n^{-s}D(s) :, n = 1, 2, \cdots\} = \mathcal{H}^2.$$

By $\mathcal{D}(\varphi(nx)) = n^{-s} \cdot \mathcal{D}\varphi$, the Beurling-Wintner problem is equivalent to the problem on cyclic vectors in $\mathcal{H}^2$, see paper by Hedenmalm, Lindqvist and Seip.

Proposition 1. Let $\varphi \in L^2$, the following statements are equivalent:

(1) the dilation system $\{\varphi(x), \varphi(2x), \cdots\}$ is complete in L^2;

(2) $\mathcal{D}\varphi$ is cyclic in $\mathcal{H}^2$;

The Beurling-Wintner transformation of step functions

Given a step function φ on $(0, 1)$. Let $J_\varphi(x)$ $(x \in \mathbb{R})$ denote the jump of φ at x:

$$J_\varphi(x) = \varphi(x^+) - \varphi(x^-) = \lim_{u \to x^+} \varphi(u) - \lim_{v \to x^-} \varphi(v),$$

where φ is identified with its odd 2-periodic extension on $\mathbb{R}$.

Now given a step function φ with rational discontinuous points, and let $s_1/t_1, \cdots, s_l/t_l$ be its all jump discontinuous points in $(0, 1)$ with $\gcd(s_i, t_i) = 1$, and let t be the least common multiple of $t_1, \cdots, t_l$. Then the step function φ can be represented as

$$\varphi = \frac{1}{2} J_\varphi(0) \chi_{(0,1)} + \sum_{m=1}^{t-1} J_\varphi(\frac{m}{t}) \chi_{(\frac{m}{t},1)}.$$

Set

$$f(n) = 2n \int_0^1 \varphi(x) \sin n\pi x \mathrm{d}x, \quad n = 1, 2, \cdots, \quad \text{and} \quad q = 2t$$

a direct calculation yields

$$\begin{aligned} f(n) &= J_\varphi(0) + J_\varphi(1) \cos n\pi + 2 \sum_{m=1}^{t-1} J_\varphi(\frac{m}{t}) \cos \frac{mn\pi}{t} \\ &= \sum_{m=1}^{q} J_\varphi(\frac{m}{t}) e^{2\pi imn/q}, \qquad \text{Gauss sum of the jump function } J_\varphi \end{aligned}$$

f is periodic with period q, that is, $f(n+q) = f(n), n = 1, 2, \cdots$.

Sine $\{\sqrt{2}\sin(n\pi x):\ n\in\mathbb{N}\}$ is a canonical orthonormal basis of L^2,

$$\varphi = 2\sum_{n=1}^{\infty}\Big[\int_0^1 \varphi(x)\sin(n\pi x)\mathrm{d}x\Big]\sin(n\pi x)$$

and hence

$$\mathcal{D}\varphi(s) = \frac{\sqrt{2}}{2}\sum_{n}\frac{f(n)}{n^{s+1}} = \frac{\sqrt{2}}{2}D_f(s+1),$$

where $D_f(s)$ is defined by

$$D_f(s) := \sum_{n=1}^{\infty}\frac{f(n)}{n^s}.$$

Therefore, $\mathcal{D}\varphi$ is a translation of D_f. Notice that the above arithmetical function f is periodic with period q. Hence in light of Proposition 1, we should first establish a criterion for the cyclicity on translations of Dirichlet series with periodic coefficients. A natural entry point is Dirichlet L-functions.

4. A brief introduction for Dirichlet characters, Dirichlet L-functions and Prime number theorem

A Dirichlet character mod q is an arithmetical function χ satisfying

1) χ is completely multiplicative, that is, for any $m, n \in \mathbb{Z}$,

$$\chi(mn) = \chi(m)\chi(n);$$

2) χ is periodic with the period q(where q is the least period), that is, $\chi(n+q) = \chi(n), n \in \mathbb{Z}$.

It is easy to verify that if χ is not zero character, then if $\gcd(n, q) > 1$, then $\chi(n) = 0$; if $\gcd(n, q) = 1$, then $\chi(n) \neq 0$.

There are exactly $\phi(q)$ distinct Dirichlet characters mod q, where $\phi(q)$ is Euler's totient function– it is the number of integers k in the range $1 \leq k \leq q$, $\gcd(q, k) = 1$.

If χ is a Dirichlet character, one defines its Dirichlet L-function by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

where s is complex variable with $\mathrm{Re}(s) > 1$. By analytic continuation, this function can be extended to a meromorphic function on the whole complex plane. Dirichlet L-functions are generalizations of the Riemann zeta-function. The case $\chi(n) = 1$ for all n yields the Riemann zeta function

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots .$$

Notice that L-function of χ can be written as an Euler product:

$$L(s,\chi) = \prod_p \left(1 - \chi(p)p^{-s}\right)^{-1} \text{ for } \mathrm{Re}(s) > 1,$$

where p run over all prime numbers. Therefore each Dirichlet L-function has no zero point in $\mathrm{Re}(s) > 1$.

The generalized Riemann hypothesis asserts that for every Dirichlet L-function, any zero point s with $\mathrm{Re}(s) > 0$ lies on $\mathrm{Re}(s) = 1/2$.

Dirichlet L-functions introduced by Dirichlet play a fundamental role in the distribution of primes in an arithmetic progression. Dirichlet's theorem states that for any pair a, q of relatively prime positive integers, the arithmetic progression $a,\ a+q,\ a+2q, \cdots$ contains infinitely many primes.

The following two theorems on prime numbers are well known, for which they can be found in some books on Number Theory.

Dirichlet's theorem: For relatively prime positive integers a, q,

$$\sum_{\substack{p \equiv a \pmod q \\ p \text{ prime}}} \frac{1}{p} = \infty.$$

Prime number theorem for arithmetic progressions: Assume $\gcd(q,a) = 1$, and let $\pi(x;q,a)$ be the number of prime numbers $p \equiv a \pmod{q}$ which do not exceed x. Then

$$\pi(x;q,a) \sim \frac{x}{\phi(q)\log x}, \qquad x \to \infty.$$

Let χ be a Dirichlet character mod q. A divisor d of q is called an induced modulus for χ if $\chi(a) = 1$ for any positive integer a, $\gcd(a,q) = 1$ and $a \equiv 1 \pmod{d}$. The Dirichlet character χ is said to be primitive if it has no induced modulus less than q.

Every Dirichlet character χ can be uniquely decomposed as a product of a primitive Dirichlet character and the principle Dirichlet character mod q, where the principle Dirichlet character χ_0 mod q is defined to be $\chi_0(n) = 1$ if $\gcd(n,q) = 1$; $\chi_0(n) = 0$ if $\gcd(n,q) > 1$.

5. The cyclicity of translations of Dirichlet series with periodic coefficients

As discussed above the B-W transformation maps step functions to translations of Dirichlet series with periodic coefficients, and notice that "cyclicity of translations"$\Leftrightarrow$ "completeness of dilation systems".

For an arithmetical function f, D_f denotes the Dirichlet series

$$D_f(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

If f is bounded, then $\sigma_a(D_f) \leq 1$ and the translations of D_f, $D_f(s+1) \in \mathcal{H}^2$, and it is analytic in $\mathbb{C}_0$,

We have the following complete characterization of the cyclicity of Dirichlet series $D_f(s+1)$ in $\mathcal{H}^2$ when f is periodic.

Main Theorem [Dan-G] Let q be a positive integer, and f is periodic with period q. Then the following statements are equivalent:

(1) $D_f(s+1)$ is cyclic in $\mathcal{H}^2$;

(2) D_f has no zeros in $\mathbb{C}_1$ and $f(1) \neq 0$;

(3) D_f has a unique decomposition

$$D_f(s) = P(s)L(s,\psi)$$

for some Dirichlet polynomial P without zeros in $\mathbb{C}_1$ and P having a nonzero constant term, and some primitive Dirichlet character ψ.

In this case, the modulus q_0 of ψ in (3) divides q, and the Dirichlet polynomial P in (3) has form

$$P(s) = \sum_{d \mid \frac{q}{q_0}} \frac{(f * \mu\psi)(d)}{d^s}.$$

The proof of this theorem relies on a technical lemma on zeros of linear combinations of Dirichlet L-functions.

Lemma [Dan-G]. For a natural number q, then a linear combination of Dirichlet L-functions, $\sum_{\chi \bmod q} c_\chi L(s,\chi)$ $(c_\chi \in \mathbb{C})$ has a nonzero constant term and no zeros in $\mathbb{C}_1$ if and only if the set $\{\chi : c_\chi \neq 0\}$ is a singleton.

The proof of this lemma is considerably technical and long, for which it needs harmonic analysis in infinitely many variables, harmonic analysis on groups and Prime number theorem for arithmetic progressions, and a covering lemma, etc.

Covering Lemma [Dan-G]. Let $\{r_n\}_{n\in\mathbb{N}}$ be a sequence of positive number with all $r_n < 1$. If $\sum_{n=1}^{\infty} r_n = \infty$, then

$$\bigcup_{m=1}^{\infty} \left\{ \prod_{n=1}^{m} z_n : z_n \in B(1, r_n) \right\} = \mathbb{C} \setminus \{0\}.$$

where $B(a,r)$ denote the disk $\{z \in \mathbb{C} : |z-a| < r\}$.

6. The B-W problem for step functions

Let $\mathcal{S}_r$ denote the class of step functions on $(0, 1)$ with rational jump discontinuities, then $\mathcal{S}_r$ is a densely linear subspace of L^2. Set

$\mathcal{C} = \{\varphi \in L^2 :$ the dilation system $\{\varphi(x), \varphi(2x), \cdots\}$ is complete in $L^2\}$

Now for $\varphi \in \mathcal{S}_r$, then $\mathcal{D}\varphi(s) = D_f(s + 1)$ for some periodic arithmetical function f. Applying Main Theorem (2) we have

Theorem 1 [Dan-G] If $\varphi \in \mathcal{S}_r$, then $\varphi \in \mathcal{C}$ if and only if $\mathcal{D}\varphi$ has a nonzero constant term and no zeros in $\mathbb{C}_0$.

Example 1. Considering $\chi_{(0,r)}(s)$, since $\mathcal{D}\chi_{(0,r)}(s) = \sum_{n=1}^{\infty} \frac{1-\cos nr\pi}{n^{s+1}}$ $(r \in [0,1])$, a direct calculation yields

$$\mathcal{D}\chi_{(0,1)}(s) = 2\sqrt{2}L(s+1,\chi_2), \quad \mathcal{D}\chi_{(0,\frac{1}{2})}(s) = \sqrt{2}(1+2^{-s})L(s+1,\chi_2),$$

$$\mathcal{D}\chi_{(0,\frac{1}{3})}(s) = \frac{\sqrt{2}}{2}(1+2^{-s}+3^{-s}-6^{-s})\zeta(s+1), \quad \mathcal{D}\chi_{(0,\frac{2}{3})}(s) = \frac{3\sqrt{2}}{2}L(s+1,\chi_3),$$

where χ_2, χ_3 are principle Dirichelt character mod 2 and 3, respectively. Then we further have

$$\mathcal{D}\chi_{(\frac{1}{2},1)}(s) = \sqrt{2}(1-2^{-s})L(s+1,\chi_2),$$

$$\mathcal{D}\chi_{(\frac{1}{3},\frac{2}{3})}(s) = \sqrt{2}(1-3^{-s})L(s+1,\chi_2),$$

$$\mathcal{D}\chi_{(\frac{1}{3},1)}(s) = \frac{3\sqrt{2}}{2}(1-2^{-s})L(s+1,\chi_3).$$

Therefore when I is one of the next 6 intervals, $(0,1), (0,\frac{1}{2}), (\frac{1}{2},1), (0,\frac{2}{3}), (\frac{1}{3},1), (\frac{1}{3},\frac{2}{3})$, the Dirichlet series $\mathcal{D}\chi_I$ has a nonzero constant terms and no zeros in $\mathbb{C}_0$, this shows $\chi_I \in C$ for all these I.

Example 2. *Suppose that $p \geq 7$ is a prime, $k \geq 1$ and V is a proper, non-degenerated open subset of $(0, 1)$. If every boundary point of V has form $\frac{s}{p^k}$ for some integer s, then $\chi_V \in C$ if and only if*

$$V = (\frac{1}{p^l}, \frac{2}{p^l}) \cup (\frac{3}{p^l}, \frac{4}{p^l}) \cup \cdots \cup (\frac{p^l - 2}{p^l}, \frac{p^l - 1}{p^l})$$

or

$$V = (0, \frac{1}{p^l}) \cup (\frac{2}{p^l}, \frac{3}{p^l}) \cup \cdots \cup (\frac{p^l - 1}{p^l}, 1)$$

for some $1 \leq l \leq k$.

However, this Example fails for $p = 2, 3, 5$.

Let $\varphi \in \mathcal{S}_r$, let $s_1/t_1, \cdots, s_l/t_l$ be its all jump discontinuous points in $(0,1)$ with $\gcd(s_i, t_i) = 1$ for $i = 1, \cdots, l$, and let t be the least common multiple of $t_1, \cdots, t_l$. We apply Main Theorem (3) to $\mathcal{D}\varphi(s)$:

Theorem 2 [Dan-G] For the above φ in $\mathcal{S}_r$, set $q = 2t$ and $g(m) = J_\varphi(\frac{m}{t})$ $(m \geq 1)$. Then $\varphi \in C$ if and only if there exists a unique primitive Dirichlet character ψ mod q_0 with $q_0 \mid q$ that satisfies:

(1) $g(m) = g(\widehat{m})\psi(\frac{m}{\widehat{m}})$ for each $m \geq 1$, where $\widehat{m} = \gcd(m, q)$;

(2) the Dirichlet polynomial $\sum_{d|\frac{q}{q_0}}(g * \mu\psi)(\frac{q}{dq_0})d^{-s}$ has a nonzero constant term and no zeros in $\mathbb{C}_0$.

According to Theorem 2 we can construct a lots of step functions belonging to C. Let $d(n)$ $(n \geq 1)$ denote the number of divisors of n.

Example 3. Let ψ be a primitive Dirichlet character mod q with $q > 1$ and $\psi(-1) = 1$. For each $n \geq 1$ put

$$S_n = \sum_{\substack{m=1 \\ \gcd(m,n)=1}}^{[\frac{nq-1}{2}]} \psi(m)\chi_{(\frac{2m}{nq},1)}.$$

(1) Suppose that v is a product of some distinct primes and $\gcd(v, q) = 1$. Then we have $S_v \in C$.

(2) Suppose that u is a positive even integer, and $\{c_r : r \mid u\}$ are complex numbers satisfying that

$$|c_1| > [d(uu') - 1] \cdot \max\{|c_r| : r \mid u, r > 1\},$$

where u' is the largest divisor of u relatively prime to q. Then we have $\sum_{r|u} c_r S_r \in C$.

Example 4. Set $\psi(n) = (n \mid 5)$, the Legendre symbol mod 5. Then Example 3 (1) gives

$$S_2 = \chi_{(\frac{1}{5},\frac{3}{5})} \in \mathcal{C},$$

$$S_3 = \chi_{(\frac{2}{15},\frac{4}{15})} + \chi_{(\frac{8}{15},\frac{14}{15})} \in \mathcal{C},$$

$$S_6 = \chi_{(\frac{1}{15},\frac{7}{15})} + \chi_{(\frac{11}{15},\frac{13}{15})} \in \mathcal{C}.$$

Moreover, by letting $u = 2$ in Example 3 (2), we conclude that

$$c_1 S_1 + c_2 S_2 = c_1 \chi_{(\frac{2}{5},\frac{4}{5})} + c_2 \chi_{(\frac{1}{5},\frac{3}{5})} \in \mathcal{C}$$

for $c_1, c_2 \in \mathbb{C}$ with $|c_1| > 2|c_2|$. In particular, $S_1 = \chi_{(\frac{2}{5},\frac{4}{5})} \in \mathcal{C}$.

7. Jumps and discontinuities for step functions in C

Applying the above Theorem 2 to a step function $\varphi \in C$ with rational jump discontinuities, we have

$$J_\varphi(\frac{m}{n}) = \begin{cases} J_\varphi(\frac{1}{n}), & m\, odd; \\ J_\varphi(\frac{2}{n})\psi(\frac{m}{2}), & m\, even \end{cases}$$

If $J_\varphi(\frac{m}{n}) \neq 0$, then $q_0|2n$ if n is even; $q_0|n$ if n is odd.

The above statement shows that jumps and discontinuities for step functions in $C \cap \mathcal{S}_r$ follow certain rule.

Theorem 3 [Dan-G]. *If $\varphi \in \mathcal{S}_r \cap C$, then the set of all jump discontinuities of φ in $(0,1)$ is*

$$\bigsqcup_{\substack{n\geq 3\\ J_\varphi(\frac{2}{n})\neq 0}} \Big\{\frac{2m}{n} : 1 \leq m < \frac{n}{2}, \gcd(m,n) = 1\Big\},$$

and

$$\sum_{\substack{n\geq 3\\ J_\varphi(\frac{2}{n})\neq 0}} \phi(n) = 2N,$$

where N is the number of jump discontinuities of φ in $(0,1)$.

Set

$$t(n) = \prod_{\substack{p \le 2n+1 \\ p \text{ prime}}} p^{[\log_p 3n]}, \quad n = 1, 2, \cdots .$$

Corollary 2 [Dan-G]. If $\varphi \in \mathcal{S}_r \cap C$ has N $(N \in \mathbb{N})$ jump discontinuities in $(0, 1)$, then every jump discontinuity of φ has form $\frac{s}{t(N)}$ for some integer s.

Therefore, functions in $\mathcal{S}_r \cap C$ satisfy very strong constraints. This results in that the set $\mathcal{S}_r \cap C$ is very "small" in some sense.

8. The B-W problem for characteristic functions

We apply the preceding results to characteristic functions on open subsets of $(0, 1)$.

Theorem 4 [Dan-G]. For each natural number n, writing $\mathcal{F}_n$ to be the set of all non-degenerated open subsets V of $(0, 1)$ with rational boundary points, which have at most n connected components. Then the set

$$\mathcal{S}_n = \{V : V \in \mathcal{F}_n, \text{and } \chi_V \in C\}$$

is a finite set. A set V is non-degenerated if $(0, 1) \setminus V$ has not isolated points.

The case $n = 1$ is especially important, and in this case we can find out all open subintervals in $\mathcal{S}_1$.

Theorem 5 [Dan-G]. *Let $0 \leq \alpha < \beta \leq 1$ be two rational numbers and put $I = (\alpha, \beta)$. Then $\chi_I \in C$ if and only if I is one of the next 10 intervals*

$$(0,1), (0,\frac{1}{2}), (\frac{1}{2},1), (0,\frac{2}{3}), (\frac{1}{3},\frac{2}{3}), (\frac{1}{3},1), (\frac{1}{4},\frac{3}{4}), (\frac{1}{5},\frac{3}{5}), (\frac{2}{5},\frac{4}{5}), (\frac{1}{6},\frac{5}{6}).$$

In particular, we have solved the rational version of the Kolzov completeness problem in 1950s:

Suppose that $r \in (0,1]$ is rational. Then $\chi_{(0,r)} \in C$ if and only if $r = 1, \frac{1}{2}, \frac{2}{3}$.

Recall that the Kozlov completeness problem is to characterize the set

$$\mathcal{K} := \{t \in (0,1] : \chi_{(0,t)} \in C\}.$$

Applying the above Theorem we conclude that for any rational number $r \in (0,1)$ other than $\frac{1}{2}, \frac{2}{3}, 1$, there exists a positive number $\varepsilon = \varepsilon(r)$, such that $\chi_{(0,t)} \notin C$ for each $t \in (0,1)$ with $|t - r| < \varepsilon(r)$. Therefore, $\mathcal{K}$ is nowhere dense in $(0,1]$, and it is reasonable to conjecture that $\mathcal{K}$ is "small" in $(0,1]$. We record the following interesting question.

Question. Is $\mathcal{K}$ of zero Lebesgue measure? Is $\mathcal{K}$ a countable set? Is $\mathcal{K}$ a finite set? Or, are $1, \frac{1}{2}, \frac{2}{3}$ the only elements in $\mathcal{K}$?

9. The B-W problem for real step functions

Theorem 6 [Dan-G]. Suppose $\varphi \in \mathcal{S}_r \cap C$ is real-valued.

(1) If $\varphi(0^+) > 0$, then for any N ($N \in \mathbb{N}$) distinct primes $p_1, \cdots, p_N$,

$$2\varphi(0^+) > \sum_{1\le i\le N} J_\varphi(\frac{2}{p_i}) - \sum_{1\le i<j\le N} J_\varphi(\frac{2}{p_i p_j}) + \cdots + (-1)^{N+1} J_\varphi(\frac{2}{\prod_{i=1}^N p_i}).$$

(2) If $\varphi(0^+) = 0$, then either $J_\varphi(\frac{2}{p}) \ge 0$ for any prime p or $J_\varphi(\frac{2}{p}) \le 0$ for any prime p.

(3) Let p be a prime with $p \equiv 3 \pmod 4$ and k a positive integer. Then $J_\varphi(\frac{m}{p^k}) = J_\varphi(\frac{n}{p^k})$ for any $1 \le m, n \le p^k - 1$ satisfying that $p \nmid mn$ and $m - n$ is even.

Notice that when $\varphi(0^+) < 0$, then "$>$" in the conclusion of the above Theorem is replaced by "$<$".

We take $N = 1$ and $p_1 = 2$ in the above Theorem, then

If $\varphi \in \mathcal{S}_r$ is real-valued and $\frac{\varphi(1^-)}{\varphi(0^+)} \leq -1$ $(\varphi(0^+) \neq 0)$, then $\varphi \notin C$.

For example, taking $\varphi = \chi_{(0,1/2)} - a\chi_{(1/2,1)}$, $a \geq 1$, then the dilation system $\{\varphi(x), \varphi(2x), \cdots\}$ of φ is incomplete in L^2.

To all interested readers, Please see our paper attached in Arxiv, The solutions to the Wintner-Beurling problem in the class of step functions, http://arxiv.org/abs/2005.09779v2.

Thank you for your attention!