Do some closed nontrivial *z*-invariant subspaces have the division property?

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Let $\mathcal{H}(\mathbb{D})$ be the (Fréchet) space of holomorphic functions on the unit disc $\mathbb{D}$, and set $\widehat{f}(n) = \frac{f^{(n)}(0)}{n!}$ for $f \in \mathcal{H}(\mathbb{D})$, $n \geq 0$. The shift S and the backward shift T are defined for $f \in \mathcal{H}(\mathbb{D})$ by the formulae
$Sf(z) = zf(z)$ $(z \in \mathbb{D})$, $Tf(z) = \frac{f(z)-f(0)}{z}$ $(z \in \mathbb{D} \setminus \{0\})$, $Tf(0) = f'(0)$, so that
$TS = I$, $Spec(S) = \mathbb{D}$ and $Spec(T) = \overline{\mathbb{D}}$.
Set $f_\lambda(z) = \frac{f(z)-f(\lambda)}{z-\lambda}$ for $z \neq \lambda$, $f_\lambda(\lambda) = f'(\lambda)$. Then $f_\lambda \in \mathcal{H}(\mathbb{D})$ for $f \in \mathcal{H}(\mathbb{D})$, $\lambda \in \mathbb{D}$, and

$$f_\lambda = T(I - \lambda T)^{-1} f.$$

Definition of spaces of holomorphic functions on $\mathbb{D}$

A space of holomorphic functions on the unit disc $\mathbb{D}$ is a (nonzero) linear space $E \subset \mathcal{H}(\mathbb{D})$ which satisfies the following properties
(i) $S(E) \cup T(E) \subset E$.
(ii) $Spec(S_{|_E}) \cup Spec(T_{|_E}) \subset \overline{\mathbb{D}}$.
A topological space of holomorphic functions on $\mathbb{D}$ is a linear space E of holomorphic functions on $\mathbb{D}$ endowed with a linear space topology for which the inclusion map $E \subset \mathcal{H}(\mathbb{D})$ is continuous.

Since $T^n f - ST^{n+1} f = \widehat{f}(n).1$ for $f \in \mathcal{H}(\mathbb{D})$, every space of holomorphic functions on $\mathbb{D}$ contains the polynomials. Also since $f(\lambda).1 = (I_E - ST)(I_E - \lambda T)^{-1} f$ for $\lambda \in \mathbb{D}, f \in \mathcal{H}(\mathbb{D})$, the inclusion map $E \subset \mathcal{H}(\mathbb{D})$ is continuous for every space of holomorphic functions on $\mathbb{D}$ which is a Fréchet space with respect to a topology for which the restriction maps $S_{|_E} : E \to E$ and $T_{|_E} : E \to E$ are continuous. Conversely it follows from the closed graph theorem that if E is a Fréchet space of holomorphic functions on $\mathbb{D}$ according to the definition then the restriction maps $S_{|_E} : E \to E$ and $T_{|_E} : E \to E$ are continuous.
For $f \in \mathcal{H}(\mathbb{D})$ set $Z(f) := \{z \in \mathbb{D} \mid f(z) = 0\}$, and for $M \subset \mathcal{H}(\mathbb{D})$ set $Z(M) := \cap_{f \in M} Z(M)$.

z-invariant subspaces and the division property

(i) A linear subspace M of $\mathcal{H}(\mathbb{D})$ is said to be z-invariant if $S(M) \subset M$.
(ii) A linear subspace M of $\mathcal{H}(\mathbb{D})$ is said to have the division property if $f_\lambda \in M$ for every $\lambda \in \mathbb{D}$ and every $f \in M$ such that $f(\lambda) = 0$.

Proposition

A linear subspace $M \neq \{0\}$ of $\mathcal{H}(\mathbb{D})$ has the division property if and only if $Z(M) = \emptyset$ and $\dim(M/((S - \lambda I)(M) \cap M)) = 1$ for every $\lambda \in \mathbb{D}$.

For Banach spaces of holomorphic functions the situation is simpler

Proposition

A Banach space of holomorphic functions on $\mathbb{D}$ has the division property if and only if $Z(M) = \emptyset$ and $\dim(M/((S - \lambda I)(M) \cap M)) = 1$ for some $\lambda \in \mathbb{D}$.

Problem 1

Does every Banach space of holomorphic functions on $\mathbb{D}$ possess a nontrivial closed z-invariant subspace having the division property?

The example of $\mathcal{H}(\mathbb{D})$ shows that the answer is negative in the Féchet space context: $\mathcal{H}(\mathbb{D})$ is a Fréchet algebra, characters of $\mathcal{H}(\mathbb{D})$ are evaluation maps $\delta_\lambda : f \to f(\lambda)$ where $\lambda \in \mathbb{D}$, and closed z-invariant subspaces of $\mathcal{H}(\mathbb{D})$ are closed ideals since polynomials are dense in $\mathcal{H}(\mathbb{D})$. So if M is a nontrivial closed z-invariant of $\mathcal{H}(\mathbb{D})$, M is contained in the kernel of a character of $\mathcal{H}(\mathbb{D})$ by a standard result of Michael [26], $Z(M) \neq \emptyset$, and M does not have the division property.
Notice that $Spec(S) = \mathbb{D}$ and $Spec(T) = \overline{\mathbb{D}}$, while if E a Banach space of holomorphic functions on $\mathbb{D}$ then $Spec(S_{|_E}) = Spec(T_{|_E}) = \overline{\mathbb{D}}$.

Clearly, the closure of a z-invariant subspace is also z-invariant. We also have the following observation

Proposition

If a linear subspace M of a Banach space of holomorphic functions on $\mathbb{D}$ has the division property, then its closure $\overline{M}$ also has the division property.

To prove our observation, notice that if $M \neq \{0\}$ has the division property then for every $\lambda \in \mathbb{D}$ there exists $u_\lambda \in M$ such that $u_\lambda(\lambda) = 1$. Now let $f \in \overline{M}$ such that $f(\lambda) = 0$, and let $(f_n)_{n\geq 1}$ be a sequence of elements of M converging to f. Then $\lim_{n\to+\infty} f_n(\lambda) = 0$. Set $g_n = f_n - f_n(\lambda)u_\lambda$. Then $g_n(\lambda) = 0$, $T_\lambda g_n \in M$, and $T_\lambda f = \lim_{n\to+\infty} T_\lambda(g_n) \in \overline{M}$ which shows that $\overline{M}$ has the division property. $\square$

The index $ind(M)$ of a z-invariant subspace M of E is by definition the dimension (finite or infinite) of the quotient space E/M, and so Problem 1 is equivalent to the following question.

Question 1

Does every Banach space of holomorphic functions on $\mathbb{D}$ possess a closed nontrivial z-invariant subspace M of index 1 such that $Z(M) = \emptyset$?

Let E be a Banach space of holomorphic functions on $\mathbb{D}$. Then E contains $\mathcal{H}(\overline{\mathbb{D}})$, the algebra of germs of holomorphic functions on $\overline{\mathbb{D}}$.

Examples

1) If polynomials are not dense in E then $\overline{\mathbb{C}(z)}^E$ is a nontrivial z-invariant subspace of E having the division property (which contains $\mathcal{H}(\overline{\mathbb{D}})$).
2) Denote by $[f]$ the set $\{g = fp : p \in \mathbb{C}[z]\}$. Then $[f]$ is z-invariant, and if $Z(f) := \{\lambda \in \mathbb{D} \mid f(\lambda) = 0\} = \emptyset$, then $[f]$ has the division property.
So if there exists a function f in E which is not z-cyclic and has no zeroes in $\mathbb{D}$, then $\overline{[f]}^E$ *is a nontrivial* z*-invariant subspace having the division property.*
3) If $\liminf_{|\lambda|\to 1^-} \|\delta_\lambda\|_{E^*} < +\infty$, there exists $\zeta \in \mathbb{T}$ and $\phi \in E^*$ such that $\phi(1) = 1$, $\phi \circ S = \zeta\phi$, and $Ker(\phi)$ *is a nontrivial* z*-invariant subspace of* E *having the division property*.

To see that 3) holds notice that there exists a sequence $(\lambda_n)_{n\geq 1}$ of elements of $\mathbb{D}$ and $\zeta \in \mathbb{T}$ such that $\lim_{n\to+\infty} \lambda_n = \zeta$ and $\sup_{n\geq 1} \|\delta_{\lambda_n}\| < +\infty$. Clearly, any cluster point of the sequence $(\delta_{\lambda_n})_{n\geq 1}$ in E^* equipped with the w^*-topology has the required properties.
It follows from (3) that if E is a Banach algebra then E has a nontrivial closed z-invariant subspace having the division property.

The author does not know any example of nontrivial closed z-invariant subspace M having the division property which does not have the form $M = [f]^{-E}$ where f is a non z-cyclic function without zeroes in $\mathbb{D}$.
Borichev, Hedenmalm and Volberg showed in [8] that such non z-cyclic functions with no zeroes in $\mathbb{D}$ do exist in "large" Bergman spaces on the disc.
The existence of noncyclic functions without zeroes for some other classes of spaces of holomorphic functions in the disc follows from works of Atzmon [2] based on the theory of entire functions and from works of Nikolski [28] based on the so-called "abstract Keldysh method", see [18], section 5.
On the other hand consider a Banach space of holomorphic functions on $\mathbb{D}$ of the form $H^p_w(\mathbb{D}) := \{f \in \mathcal{H}(\mathbb{D}) \mid \sum_{n=0}^{+\infty} |\widehat{f}(n)|^p \omega(n)^p < +\infty\}$, $1 \leq p < +\infty$, where ω is a weight on $\mathbb{Z}^+$. Borichev showed in [6] that if $\liminf_{n\to+\infty} \omega(n) = 0$, then $H^p_w(\mathbb{D})$ possesses closed zero-free z-invariant subspaces of arbitrary index $m \in \{2, 3, \dots, \infty\}$, but, unfortunately, his construction does not work for $m = 1$.
We refer more generally to Section 5 of [18] for a detailed discussion of the existence of nontrivial closed z-invariant subsbaces having the discussion property for weighted H^2-spaces of holomorphic functions on $\mathbb{D}$.

If $\|f\|_E \leq \|g\|_E$ for every $f, g \in E$ such that $|f(\zeta)| \leq |g(\zeta)|$ for every $\zeta \in \mathbb{D}$, then $\|p(S)\| \leq \max_{\zeta \in \mathbb{T}} |p(\zeta)|$ for every polynomial p, the shift S satisfies the von Neumann inequality, and the Brown-Chevreau-Pearcy theory [3], [12] and its extensions to general Banach spaces [1] apply and produce a very rich lattice of closed z-invariant subspaces of E (we are in the "easy" situation where the spectrum of S equals the whole of $\mathbb{D}$). The author was not able so far to use these methods to produce zero-free non z-cyclic elements of E or more generally closed nontrivial z-invariant subspaces of E having the division property in this situation.
In the opposite direction it does not seem that Read's method construction of a counterexample to the invariant subspace problem [30] and its adaptation to Hilbert spaces [20] gives any clue to construct a counterexample to Problem 1.

Let E be a Banach space of holomorphic functions on $\mathbb{D}$, and assume that there exists a Banach space F isometrically containing E such that S has a bounded extension $\mathbf{S}$ to F. Clearly, if M is a z-invariant subspace of E, and if $\mathbf{S}$ is invertible, then $\overline{[\cup_{n\geq 1}\mathbf{S^{-n}(M)}]}^F$ is a closed invariant subspace of F which is invariant for $\mathbf{S}$ and $\mathbf{S^{-1}}$.The motivation for the study of z-invariant subspaces of E having the division property follows from the following observation. Here $\mathbb{T}$ denotes the unit circle.

Right-invariant subspaces and the division property

Proposition

Let N be a closed subspace of F. If Spec$(\mathbf{S}) \subset \mathbb{T}$, and if N is invariant for $\mathbf{S^{-1}}$, then $N \cap E$ is a closed subspace of E having the division property.

To see this notice that if $\lambda \in \mathbb{D}$, then $(\mathbf{S} - \lambda I_F)^{-1} = \mathbf{S}^{-1}(I_F - \lambda \mathbf{S}^{-1})^{-1} = \sum_{n=1}^{+\infty} \lambda^n \mathbf{S}^{-n-1}$, and so $(\mathbf{S} - \lambda I_F)^{-1} f \in N$ if N is invariant with respect to $\mathbf{S}^{-1}$. Now if $f \in N \cap E$, then $(S - \lambda I_E)T_\lambda f = f - f(\lambda).1$. So $T_\lambda f = (\mathbf{S} - \lambda I_F)^{-1}(f - f(\lambda).1) \in E$, and $T_\lambda f = (\mathbf{S} - \lambda I_F)^{-1} f \in N \cap E$ if $f(\lambda) = 0$. $\square$

Consider again a Banach space E of holomorphic functions on $\mathbb{D}$, and a Banach space F as above. There are two natural questions.

Question 2

Find sufficient conditions insuring that $\overline{[\cup_{n\geq 0}\mathbf{S}^{-n}(M)]}^F \cap E = M$, *where M is a nontrivial closed z-invariant subspace of E having the division property.*

Question 3

Find sufficient conditions which ensure that every nontrivial closed subspace of F invariant for $\mathbf{S}$ and $\mathbf{S}^{-1}$ has the form $\overline{[\cup_{n\geq 0}\mathbf{S}^{-n}(M)]}^F$, *where M is a closed z-invariant subspace of E having the division property.*

Closed subspaces of F invariant for $\mathbf{S}$ and $\mathbf{S}^{-1}$ and closed z-invariant subspaces of E having the division property may be unrelated, see [32] and Beurling's theorem on $H^2(\mathbb{D})$.

Example

Set $E = H^2(\mathbb{D}) \approx H^2(\mathbb{T})$, $F = L^2(\mathbb{T})$, *and* $\mathbf{S}(f)(\zeta) = \zeta f(\zeta)$ *on* $\mathbb{T}$ $(f \in L^2(\mathbb{T}))$. *Then* $N \cap E = \{0\}$ *for every nontrivial closed subspace N of F invariant for $\mathbf{S}$ and $\mathbf{S}^{-1}$, and* $\cup_{n\geq 1}\mathbf{S}^{-1}(M)$ *is dense in F for every nonzero closed z-invariant subspace M of E having the division property.*

Let $\mathcal{H}_0(\mathbb{C} \setminus \overline{\mathbb{D}})$ be the space of holomorphic functions on $\mathbb{C} \setminus \overline{\mathbb{D}}$ vanishing at infinity, and denote by $\mathcal{HF}(\mathbb{T})$ the space of hyperfunctions on $\mathbb{T}$, i.e. the space of all pairs $f = (f^+, f^-)$ where $f^+ \in \mathcal{H}(\mathbb{D})$ and $f^- \in \mathcal{H}_0(\mathbb{C} \setminus \overline{\mathbb{D}})$. Hyperfunctions on the circle form a flabby sheaf [11], the notion of support of a distribution can be extended to hyperfunctions, the "product" of two hyperfunctions with disjoint support vanishes in some natural sense [17] and several variables extensions of the notion of hyperfunction play a basic role in microlocal calculus. We will not use these saddleties here, but we will need standard properties of the topological convex linear space $\mathcal{HF}(\mathbb{T})$.
As a product of Fréchet spaces, $\mathcal{HF}(\mathbb{T})$ is a Fréchet space. Identifying g with $(g, 0)$ for $g \in \mathcal{H}(\mathbb{D})$, we can consider $\mathcal{H}(\mathbb{D})$ as a closed subspace of $\mathcal{HF}(\mathbb{T})$. Since products of nuclear Fréchet spaces are nuclear, the fact that $\mathcal{H}(\mathcal{U})$ is nuclear for every open subset $\mathcal{U}$ of $\mathbb{C}$, see [29], theorem 6.4.2, implies that $\mathcal{HF}(\mathbb{T})$ is a nuclear Fréchet space.

For $s \in (0,1)$ set $\mathcal{U}_s := \{\lambda \in \mathbb{C} \ : \ s \le |\lambda| < s^{-1}\}$, and let $\mathcal{O}(\mathbb{T}) := \cup_{s\in(0,1)}\mathcal{H}(\mathcal{U}_s)$ be the space of germs of analytic functions on $\mathbb{T}$, equipped with the usual inductive limit topology. One can identify $\mathcal{O}(\mathbb{T})$ with the dual space of $\mathcal{HF}(\mathbb{T})$, equipped with the topology of uniform convergence on bounded subsets of $\mathcal{HF}(\mathbb{T})$, by using for $h \in \mathcal{H}(\mathcal{U}_s), 0 < s < 1$ the formula

$$< f, h >= \frac{1}{2i\pi}\int_{r\mathbb{T}} f(\zeta)h(\zeta)d\zeta + \frac{1}{2i\pi}\int_{R\mathbb{T}} f(\zeta)h(\zeta)d\zeta \quad (f \in \mathcal{HF}(\mathbb{T})),$$

where $s < r < 1 < R < s^{-1}$, and where the unit circle $\mathbb{T}$ is oriented counterclockwise, see the details in Chapter 1 of [4].

The Fourier coefficients $\widehat{f}(n)$ and $\widehat{h}(n)$ for $f \in \mathcal{HF}(\mathbb{T})$ and $h \in \mathcal{H}(\mathcal{U}_s)$ are defined by the formulae $f^+(\zeta) = \sum_{n=0}^{+\infty} \widehat{f}(n) \ (|\zeta| < 1), \ f^-(\zeta) = \sum_{n<0} \widehat{f}(n)\zeta^n \ (|\zeta| > 1)$, $h(\zeta) = \sum_{n=-\infty}^{+\infty} \widehat{h}(n)\zeta^n \ (\zeta \in \mathcal{U}_s)$, which gives

$$< f, h >= \sum_{n\in\mathbb{Z}} \widehat{f}(n)\widehat{h}(-n-1).$$

It follows from reflexivity properties of nuclear Fréchet spaces [29], Th. 4.4.13 that the duality formula identifes $\mathcal{HF}(\mathbb{T})$ to the dual space of $\mathcal{O}(\mathbb{T})$, equipped with the topology of uniform convergence on bounded subsets of $\mathcal{O}(\mathbb{T})$.

The (bilateral) shift operator **S** on $\mathcal{HF}(\mathbb{T})$ is defined by the formula

$$\left(\widehat{\mathbf{S}f}\right)(n) = \widehat{f}(n-1) \quad (f \in \mathcal{HF}(\mathbb{T}), n \in \mathbb{Z}),$$

so that $Spec(\mathbf{S}) = \mathbb{T}$.

Definition of spaces of hyperfunctions on $\mathbb{T}$

Denote by $P^+ : (f^+, f^-) \to f^+$ and $P^- : (f^+, f^-) \to f^-$ the projection maps from $\mathcal{HF}(\mathbb{T})$ onto $\mathcal{H}(\mathbb{D})$ and $\mathcal{H}_0(\mathbb{C} \setminus \overline{\mathbb{D}})$.
A space of hyperfunctions on the unit circle $\mathbb{T}$ is a (nonzero) linear space $F \subset \mathcal{HF}(\mathbb{T})$ which satisfies the following properties
(i) $P^+(F) \cup P^-(F) \subset F$,
(ii) $\mathbf{S}(F) \subset F$, and $Spec(\mathbf{S}_{|_F}) \subset \mathbb{T}$.
A topological space of hyperfunctions on $\mathbb{T}$ is a linear space F of hyperfunctions on $\mathbb{T}$ endowed with a linear space topology for which the inclusion map $F \subset \mathcal{HF}(\mathbb{T})$ is continuous.

We see as above that a Banach space of hyperfunctions on $\mathbb{T}$ always contains $\mathcal{O}(T)$.

For $g \in \mathcal{H}_0(\mathbb{C} \setminus \overline{\mathbb{D}})$, $|\zeta| > 1$, set $S^{-}g(\zeta) = \zeta g(\zeta) - \widehat{f}(-1)$, $T^{-}g(\zeta) = \frac{g(\zeta)}{\zeta}$. We define as above a space of holomorphic functions on $\mathbb{C} \setminus \overline{\mathbb{D}}$ vanishing at infinity to be a space $E^{-} \subset \mathcal{H}_0(\mathbb{C} \setminus \overline{\mathbb{D}})$ such that $S^{-}(E^{-}) \cup T^{-}(E^{-}) \subset E^{-}$ and $Spec(S^{-}_{|_{E^{-}}}) \cup Spec(S^{-}_{|_{E^{-}}}) \subset \mathbb{D}$, and we define a topologial space of holomorphic functions on $\mathbb{C} \setminus \overline{\mathbb{D}}$ vanishing at infinity to be a space of holomorphic functions on $\mathbb{C} \setminus \overline{\mathbb{D}}$ vanishing at infinity E^{-} endowed with a linear space topology with respect to which the inclusion $E^{-} \subset \mathcal{H}_0(\mathbb{C} \setminus \overline{\mathbb{D}})$ is continuous.

Projections

Denote by $\mathcal{E}$ (resp $\mathcal{E}^{-}$) the class of Banach spaces of holomomorphic functions on $\mathbb{D}$ (resp. on $\mathbb{C} \setminus \overline{\mathbb{D}}$ vanishing at infinity), and denote by $\mathcal{F}$ the class of Banach spaces of hyperfunctions on $\mathbb{T}$.

Proposition

If $F \in \mathcal{F}$, then $P^{+}(F) \in \mathcal{E}$ and $P^{-}(F) \in \mathcal{E}^{-}$.
Conversely if $E \in \mathcal{E}$, and if $E^{-} \in \mathcal{E}-$, then $E \oplus E^{-} \in \mathcal{F}$.

Consider a decreasing weight ω on the negative integers such that the space $H^2_{0,\omega}(\mathbb{C}\setminus\overline{\mathbb{D}}) := \{f \in \mathcal{H}_0(\mathbb{C}\setminus\overline{\mathbb{D}}) \mid \sum_{n=0}^{+\infty} |\widehat{f}(n)|^2 < +\infty\} \in \mathcal{E}^-$.

The following results, based on the growth of the inverse of singular innner functions as $|z| \to 1^-$, and on the fact that this growth may be arbitrarily slow, were proved in [14].

Answers to Question 2 for dissymmetric weighted shifts

Theorem

Denote by M_f the closed subspace of $H^2(\mathbb{D}) \oplus H^2_{0,\omega}(\mathbb{C}\setminus\overline{\mathbb{D}})$ spanned by $\{\mathbf{S}^n f :_{n\in\mathbb{Z}}\}$ for $f \in H^2(\mathbb{D})$.

(i) If $\lim_{n\to+\infty} \omega(-n) = +\infty$, there always exists a singular inner function U such that $M_U \cap H^2(\mathbb{D}) = UH^2(\mathbb{D})$.

(ii) if $\lim_{n\to-\infty} \frac{log\omega(-n)}{\sqrt{n}} = +\infty$, then $M_U \cap H^2(\mathbb{T}) = UH^2(\mathbb{T})$ for every singular inner function U.

The space $H^2(\mathbb{D}) \oplus H^2_{0,\omega}(\mathbb{C}\setminus\overline{\mathbb{D}})$ is isomorphic to the weighted L^2-space $L^2_\omega(\mathbb{T}) := \{f \in L^2(\mathbb{T}) \mid \sum_{n<0} |\widehat{f}(n)|^2\omega(n)^2 < +\infty\}$.

If $\sum_{n<0} \frac{log\omega(-n)}{n^2} < +\infty$, it follows from the discrete version of the Beurling Malliavin theorem that the space $N_L := \{f \in L^2_\omega(\mathbb{T}) \mid f(\zeta) = 0 \text{ a.e. on } L\}$ is a nontrivial closed subspace of $L^2_\omega(\mathbb{T}) \approx H^2(\mathbb{D}) \oplus H^2_{0,\omega}(\mathbb{C} \setminus \overline{\mathbb{D}})$ invariant for **S** and $\mathbf{S}^{-1}$ for every proper nonempty arc $L \subset \mathbb{T}$. So when $\lim_{n\to+\infty} log\omega(-n) = +\infty$, there are two unrelated families of "doubly invariant subspaces" and no classification of doubly invariant subspaces is known in such situations.

Theorem

Let ω be a weight on the negative integers satisfying the following conditions
(i) $\sum_{n<0} \frac{log(\omega(n))}{n^2} = +\infty$
(ii) the sequence $\left(\frac{log(\omega(-n)}{\sqrt{n}}\right)_{n\geq 1}$ *is eventually increasing*
(iii) the sequence $\left(\frac{\omega(-n)}{n^\alpha}\right)$ *is eventually log-concave for some* $\alpha > 3/2$,
Then every closed subspace M of $H^2(\mathbb{D}) \oplus H^2_{0,\omega}(\mathbb{C} \setminus \overline{\mathbb{D}})$ *invariant for* **S** *and* $\mathbf{S}^{-1}$ *has the form* $M = M_U$ *for some singular inner function U.*

The proof uses of the theory of asymptotically holomorphic functions [9] and of the so-called Dynkin trasform, and is related to the papers [7] and [31].

If $E \in \mathcal{E}$, and if M is a subspace of E having the division property, then there exists for every $\lambda \in \mathbb{D}$ a linear map $U_\lambda : E/M \to E/M$ such that $U_\lambda \circ \pi \circ (S_{|_E} - \lambda I_E) = \pi$, where $\pi = E \to E/M$ denotes the canonical surjection, see [16], Prop. 2.4. The following result is a general Banach space version of Theorem 3.3 of [18], obtained by A. Volberg and the author in the context of weighted Hardy spaces and weighted Hilbert spaces of hyperfunctions on the circle, and gives an answer to question 2 in the general case.

Theorem

Let $F \in \mathcal{F}$, set $L_n(f) = \widehat{f}(n)$ for $f \in F$, $n \in \mathbb{Z}$, let M be a closed subspace of $F^+ := F \cap \mathcal{H}(\mathbb{D})$ having the division property at 0, let $\pi : F^+ \to F^+/M$ be the canonical surjection, and let $U_0 : F^+/M \to F^+/M$ be the linear map satisfying $U_0 \circ \pi \circ S_{|_{F^+}} = \pi$.

If $\sum_{p=1}^{+\infty} \|L_{-p}\| \|U_0^p \pi(1)\| < +\infty$, then $\overline{\left[\cup_{p \geq 0} \mathbf{S}_{|_F}^{-p}(M)\right]}^F \cap F^+ = M$, and

$$\overline{\left[\cup_{n \geq 0} \mathbf{S}_{|_F}^{-p}(F^+)\right]}^F = \overline{\left[\cup_{p \geq 0} \mathbf{S}_{|_F}^{-p}(M)\right]}^F + F^+.$$

To apply the previous theorem, we will need estimates of the growth of quotient of analytic functions, due to Matsaiev-Mogulskii [21], th.1 , in the version given in [18], corollary 4.2, see also [10], [22], [23], [24], [28], [27].

Theorem

Let $\Delta : [0,1) \to (0,+\infty)$ be a continuous increasing function and let $f \in \mathcal{H}(\mathbb{D})$. Assume that there exists $f_1, f_2 \in \mathcal{H}(\mathbb{D}) \setminus \{0\}$ such that $f_2(0) \neq 0, f_2 f = f_1$ satisfying for $i = 1, 2$ the condition

$$\limsup_{|\lambda|\to 1^-} (log|f_i(\lambda)| - \Delta(|\lambda|) < +\infty.$$

(i) If $\int_0^1 \sqrt{\frac{\Delta(t)}{1-t}}dt < +\infty$, then $log^+|f(\lambda)| = O\left(\frac{1}{1-|\lambda|}\right)$ as $|\lambda| \to 1^-$.

(ii) If $\int_0^1 \sqrt{\frac{\Delta(t)}{1-t}}dt = +\infty$, then we have, for every $\epsilon > 0$,

$$\limsup_{|\lambda|\to 1^-} (1 - |\lambda|)\, log|f(\lambda)| \left[\int_0^{|\lambda|^{\frac{1}{1+\epsilon}}} \sqrt{\frac{\Delta(t)}{1-t}}dt\right]^{-2} \leq C(\epsilon), \text{ where} \quad (1)$$

$$C(\epsilon) = \frac{54}{\pi}\epsilon^{-3}(1+\epsilon)\left(1 + \frac{2\epsilon}{3}\right)^2 \left(1 + \frac{44}{5}e^{(26\pi+3/2)(2+\epsilon^{-1})}\right). \quad (2)$$

In order to estimate the growth of the sequence $(\|U^n\pi(1)\|)_{n\geq 1}$, we will need the following two lemmas.

Lemma

Let $E \in \mathcal{E}$, let M be a closed subspace of E having the division property, let $\pi : E \to E/M$ be the canonical surjection, let $U := U_0 : E \to E/M$ be the linear map satisfying $U \circ \pi \circ S_{|_E} = \pi$, and let $f \in M \setminus \{0\}$. Then we have, for $\lambda \in \mathbb{D}$,

$$f(\lambda)U(I_{E/M} - \lambda U)^{-1}\pi(1) = -\pi(f_\lambda).$$

Lemma

Let $E \in \mathcal{E}$, set $L_0(f) = f(0)$ for $f \in E$, and set, for $r \in [0, 1)$,

$$\Delta_E(r) := \sum_{r=0}^{+\infty} r^n \|T^n_{|_E}\|.$$

Then we have, for $f \in E$, $\lambda \in \mathbb{D}$,

$$|f(\lambda)| \leq \|L_0\|\|f\|\Delta_E(|\lambda|), \ \|f_\lambda\| \leq \|T_{|_E}\|\|f\|\Delta_E(|\lambda|).$$

Using the Matsaiev-Mogulski estimates, we obtain the following result

Proposition

Let $E \in \mathcal{E}$, let M be a closed subspace of E having the division property, let $\pi : E \to E/M$ be the canonical surjection, let $U := U_0 : E \to E/M$ be the linear map satisfying $U \circ \pi \circ S_{|_E} = \pi$.

(i) If $\int_0^1 \sqrt{\frac{\Delta_E(t)}{1-t}} dt < +\infty$, then we have,

$$log^+ \left\| \sum_{n=0}^{+\infty} \lambda^n U^{n+1} \pi(1) \right\| = O\left(\frac{1}{1-|\lambda|}\right) \text{ as } |\lambda| \to 1^-, \tag{3}$$

(ii) If $\int_0^1 \sqrt{\frac{\Delta_E(t)}{1-t}} dt = +\infty$, then we have, for every $\epsilon > 0$,

$$\limsup_{|\lambda| \to 1^-} (1-|\lambda|) \, log \left\| \sum_{n=0}^{+\infty} \lambda^n U^{n+1} \pi(1) \right\| \left[\int_0^{|\lambda|^{\frac{1}{1+\epsilon}}} \sqrt{\frac{\Delta_E(t)}{1-t}} dt \right]^{-2} \leq C(\epsilon), \tag{4}$$

With the notations of the Proposition, this gives the following result

Corollary

(i) If $\int_0^1 \sqrt{\frac{\Delta_E(t)}{1-t}}dt < +\infty$, *then we have,*

$$log^+\|U^n\pi(1)\| = O\left(\frac{1}{\sqrt{n}}\right) \text{ as } n \to +\infty. \quad (5)$$

(ii) If $\int_0^1 \sqrt{\frac{\Delta_E(t)}{1-t}}dt = +\infty$, *then we have, for every* $\epsilon > 0$, *when n is sufficiently large,*

$$\|U^n\pi(1)\| \leq \inf_{0<r<1} r^{-n}e^{L_{\epsilon,E}(r)}. \quad (6)$$

where $C(\epsilon)$ *is given by (2), and where* $L_{\epsilon,E}(r) = \frac{C(\epsilon)+1}{1-r}\left[\int_0^{r^{\frac{1}{1+\epsilon}}} \sqrt{\frac{\Delta_E(t)}{1-t}}dt\right]^2$.

Denote by $\mathcal{S}^-$ the set of positive weights on the negative integers such that $0 < \inf_{n\geq 1} \frac{\omega(-n)}{\omega(-n-1)} < \sup_{n\geq 1} \frac{\omega(-n)}{\omega(-n-1)}$ and $\lim_{n\to+\infty} \tilde{\sigma}(m)^{\frac{1}{m}} = \lim_{n\to+\infty} \overline{\sigma}(m)^{\frac{1}{m}} = 1$, where $\tilde{\sigma}(m) = \inf_{n\geq 1} \frac{\omega(-n)}{\omega(-n-m)}$ and $\overline{\sigma}(m) = \sup_{n\geq 1} \frac{\omega(-n)}{\omega(-n-m)}$. These conditions ensure that the space $H^p_{0,\sigma} := \{f \in \mathcal{H}_0(\mathbb{C} \setminus \overline{\mathbb{D}}) \mid \sum_{n<0} |\widehat{f}(n)|^p \omega(n)^p < +\infty\}$ is a Banach space of holomorphic functions vanishing at infinity on $\mathbb{C} \setminus \overline{\mathbb{D}}$.

Answers to question 2

Let $E \in \mathcal{E}$, and assume that $log\| T^n_{|E} \| = O(n^\alpha)$ as $n \to +\infty$, where $\alpha \in [0, 1)$. Let $\sigma \in \mathcal{S}^-$. It follows from the above discussion and from computations given in [18], section 4, that that every closed subspace of E having the division property satisfies the condition of question 2 with respect to $F = E \oplus H^p_\sigma(\mathbb{C} \setminus \overline{\mathbb{D}})$, $1 \leq p < +\infty$ if σ satisfies the following conditions

- $\liminf_{n\to+\infty} \frac{log(\sigma(-n))}{\sqrt{n}} = +\infty$ when $0 \leq \alpha < \frac{1}{2}$
- $\liminf_{n\to+\infty} \frac{log(\sigma(-n))}{\sqrt{n}log(n+1)} = +\infty$ when $\alpha = \frac{1}{2}$
- $\liminf_{n\to+\infty} \frac{log(\sigma(-n))}{n^\alpha} = +\infty$ when $\frac{1}{2} < \alpha < 1$.

Let F be a Banach space of hyperfunctions on $\mathbb{T}$. We will say that a subspace M of F is left-invariant if $\mathbf{S}^{-1}(M) \subset M$, and we will say that a left-invariant subspace M of F is analytic if $M \cap \mathcal{H}(\mathbb{D}) \neq \{0\}$. The following result is an extension to general Banach spaces of holomorphic functions on $\mathbb{D}$ of a result obtained in [19] for weighted H^2-spaces.

Theorem

Let $E \in \mathcal{E}$, and assume that a weight $\sigma \in \mathcal{S}^-$ satisfies the following conditions
(i) the sequence $\left(\frac{\sigma(-n)}{n^\alpha}\right)_{n\geq 1}$ is eventually log-concave for some $\alpha > 3/2$,
(ii) the sequence $\left(\frac{\log(\sigma(-n))}{n} log(n)^A\right)_{n\geq 1}$ is eventually nondecreasing for some $A > 0$,
(iii) $\sum_{n=1}^{+\infty} \frac{log(\sigma(-n))}{n^2} = +\infty$,
(iv) $\limsup_{n\to+\infty} \frac{log\|T^n_{|E}\|}{log(\sigma(-n))} < \frac{1}{200}$.
Then for every closed left-invariant subspace $N \neq \{0\}$ of $F = E \oplus H^2_{0,\sigma}$ there exists $k \geq 0$ such that $\mathbf{S}^k(N)$ is analytic, and we have $F = \mathbf{S}^k(N) + E$ and $N = \left[\cup_{n\geq k}\mathbf{S}^{-n}\left(\mathbf{S}^k(N) \cap E\right)\right]^-$. In particular every nontrivial closed bi-invariant subspace N of F has the form $N = [\cup_{n\geq k}\mathbf{S}^{-n}(M \cap E)]^-$, where $M = N \cap E$ is a closed z-invariant subspace of E having the division property.

As in the case of weighted Hardy spaces on the disc, the proof of the theorem uses Dynkin's extensions [13], which associate to every function $g \in H^2_{\omega,0}(\mathbb{C} \setminus \overline{\mathbb{D}})$ a smooth function G on the open unit disc which is "asymptotically analytic" in the sense that $\frac{\partial G}{\partial \overline{z}}(\zeta) \to 0$ "quickly and regularly" as $|\zeta| \to 1^-$. The theory of asymptotically holomorphic functions developped in [5], [9] shows then that if $f \in F$ is not cyclic for $\mathbf{S}_{|_F}^{-1}$ then the sequence $(\|\mathbf{S}_{|_\mathbf{F}}{}^{-n} f\|)$ grows in a much slower way as $n \to +\infty$ than the sequence $(\|\mathbf{S}_{|_\mathbf{F}}{}^{-n}\|)$. The regularity conditions (i) and (ii) play an essential role in the proof. We refer to the preprint [16] for details.

Notice that our examples of answers to question 2 involve Banach spaces E^- of the form $E^- = H^p_{0,\omega}(\mathbb{C} \setminus \overline{\mathbb{D}})$. In fact in this cases the only relevant properties concern the rate of of decrease of the sequence $(\|L_n\|_{(E^-)^*}\|)$ as $n \to -\infty$, where $L_n(g) = \widehat{g}(n)$ for $n < 0, g \in E^-$, and no regularity conditions are needed there. So in the examples the weighted Hardy spaces $H^p_{0,\omega}(\mathbb{C} \setminus \overline{\mathbb{D}})$ may be replaced by any Banach space of holomorphic functions on $\mathbb{C} \setminus \overline{\mathbb{D}}$ vanishing at infinity for which the norms $(\|L_n\|_{(E^-)^*}\|)$ satisfy for $n < 0$ the conditions stated for the coefficients $\sigma(n)$. Stronger sufficient conditions involve the norms $(\|(S^-_{|_{E^-}})^n\|)$, since $L_n = L_1 \circ (S^-_{|_{E^-}})^{n+1}$ for $n \leq -1$.

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