Multiplicity-free products of Schubert divisors and an application to canonical dimension of torsors

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(work in progress,
preprints arXiv:1711.02058 and arXiv:2011.00950)

11 June 2021

Part 1: Schubert calculus: problem statement

Notation

G = a simple split simply connected algebraic group over an arbitrary field F,
B = a Borel subgroup of G,
T = a maximal torus in B.
$\Phi \subset \mathfrak{X}(T) \otimes \mathbb{Q}$ is the corresponding root system.
Φ^+ (Φ^-) is the set of positive (resp. negative) roots.
$\Pi = \{\alpha_1, \ldots, \alpha_r\}$ = the set of simple roots.
$\langle \cdot, \cdot \rangle$ = the Cartan number on $\mathfrak{X}(T) \otimes \mathbb{Q}$,

$$\langle v, w \rangle = \frac{2v \cdot w}{w \cdot w}$$

For *each* (not nec. simple) root α, set $\sigma_\alpha \colon \mathfrak{X}(T) \otimes \mathbb{Q} \to \mathfrak{X}(T) \otimes \mathbb{Q}$,
$\sigma_\alpha v = v - \langle v, \alpha \rangle \alpha$ (the reflection along α).
ϖ_i = the fundamental weight corresponding to α_i.
$W = N_G(T)/T$ the Weyl group.
$\ell \colon W \to \mathbb{Z}$ is the length function.
For each $w \in W$, denote by $\dot{w}$ an element of $N_G(T)$ projecting to w.
w_0 = the element of maximal length in W.

Example: $G = SL_{r+1}$

$G = SL_{r+1}(F)$,
B = upper-triangular matices with determinant 1,
T = diagonal matirces with determinant 1.
$\Phi = \{\varepsilon_i - \varepsilon_j \mid i \neq j, 1 \leq i, j \leq r+1\}$, ε_i computes the ith diagonal entry.
$\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq r+1\}$ (for Φ^-: $j < i$).
$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ $(1 \leq i \leq r)$.
For the Cartan number, the dot product on $\mathfrak{X}(T) \otimes \mathbb{Q}$ is $\varepsilon_i \cdot \varepsilon_j = \delta_{ij}$.
If $\alpha = \varepsilon_i - \varepsilon_j$, then σ_α is the premutation of the ith and the jth coordinates.
$\varpi_i = \varepsilon_1 + \ldots + \varepsilon_i$.
$W = S_{r+1}$, the permutation group.
ℓ counts inversions in S_{r+1}.

$$w_0 = \begin{pmatrix} 1 & ,2, \ldots, r, r+1 \\ r+1, r, \ldots, 2, & 1 \end{pmatrix}$$

Multiplication in $\mathrm{CH}(G/B)$

Question, roughly speaking: How to compute products in $\mathrm{CH}(G/B)$?
(Over $\mathbb{C}$, via Poincare duality: how to compute products in $H^*(G/B)$ in classical topology?)

Theorem (additive structure, known before)

$\mathrm{CH}(G/B)$ *as an abelian group is freely generated by the classes of so-called Schubert varieties, i. e. varieties of the form* $Z_w = \overline{B\dot{w}_0\dot{w}B/B}$ *for all* $w \in W$.

People often use notation $X_w = \overline{B\dot{w}B/B}$, but for us, the notation Z_w will be more convenient, because the degree of Z_w in $\mathrm{CH}(G/B)$ equals $\mathrm{codim}_{G/B} Z_w = \ell(w)$.

Facts

- $Z_1 = [G/B]$, $Z_{w_0} = [\mathrm{pt}]$.
- Suppose $w_1, w_2 \in W$ are such that $\ell(w_1) + \ell(w_2) = \dim(G/B)$. Then

$$[Z_{w_1}][Z_{w_2}] = \begin{cases} [\mathrm{pt}], & \text{if } w_2 = w_1 w_0 \\ 0, & \text{otherwise} \end{cases}$$

Multiplication in $\mathrm{CH}(G/B)$

Problem: Too many classes, hard to multiply.

For each $\alpha_i \in \Pi$, denote $D_i = Z_{\sigma_{\alpha_i}}$. These are subvarieties of codimension 1, they are called *Schubert divisors*.

Theorem (known before)

The subring of $\mathrm{CH}(G/B)$ (multiplicatively) generated by Schubert divisors is a subgroup of finite index of $\mathrm{CH}(G/B)$.

Particular question 1: Let $w \in W$, $k_1, \dots, k_r \in \mathbb{Z}_{\geq 0}$, and $\ell(w) + k_1 + \ldots + k_r = \dim(G/B)$. When is $Z_w D_1^{k_1} \dots D_r^{k_r} = [\mathrm{pt}]$?

Definition

We call a monomial in Schubert divisors $D_1^{k_1} \dots D_r^{k_r}$ *multiplicity-free* if there exists $w \in W$ such that $Z_w D_1^{k_1} \dots D_r^{k_r} = [\mathrm{pt}]$.

Monomials in Schubert divisors

General question: How to express a product of Schubert divisors as a linear combination of Schubert classes?

$$D_1^{k_1} \dots D_r^{k_r} = \sum_{w \in W} c_{w,k_1,\dots,k_r} Z_w$$

How to find $c_{w,k_1,\dots,k_r}$?

Proposition (known before)

$c_{w,k_1,\dots,k_r} \geq 0$

Particular question 2: When is $c_{w,k_1,\dots,k_r} = 0$?
Particular question 3: When is $c_{w,k_1,\dots,k_r} = 1$?

Remark

*$c_{w,k_1,\dots,k_r} = 1$ if and only if $Z_{ww_0} D_1^{k_1} \dots D_r^{k_r} = [pt]$.
So, a monomial $D_1^{k_1} \dots D_r^{k_r}$ is multiplicity free if and only if there exists $w \in W$ such that $c_{w,k_1,\dots,k_r} = 1$.*

Part 2: Schubert calculus: solution

Chevalley-Pieri formula

Theorem (Chevalley-Pieri formula)

For $\alpha_i \in \Pi$, $w \in W$ we have

$$D_i Z_w = \sum_{\substack{\alpha \in \Phi^+ \\ \ell(\sigma_\alpha w) = \ell(w)+1}} \langle \varpi_i, \alpha \rangle Z_{\sigma_\alpha w}$$

From now on, the Dynkin diagram is assumed to be simply-laced (types A, D, E), and all roots are assumed to be of length 2.
Then $\langle \varpi_i, \alpha \rangle$ is the coefficient at α_i in the decomposition of α into a linear combination of simple roots: $\alpha = \sum \langle \varpi_i, \alpha \rangle \alpha_i$.

Example ($G = SL_{r+1}$)

For a positive root $\varepsilon_i - \varepsilon_j$ ($i < j$), we have $\varepsilon_i - \varepsilon_j = \alpha_i + \ldots + \alpha_{j-1}$
If $\sigma_\alpha = (ij)$ (i.e. $\alpha = \varepsilon_i - \varepsilon_j$), then $\langle \varpi_k, \alpha \rangle = 1$ if $i \leq k < j$, 0 othw.

When a coefficient equals 0

Definition

Let $\alpha \in \Phi^+$. The *support* of α is $\operatorname{supp}\alpha = \{\alpha_i \in \Pi : \langle \varpi_i, \alpha \rangle > 0\}$.

Theorem (R. D.)

$c_{w,k_1,\ldots,k_r} > 0$ if and only if there is a function $f \colon \Phi^+ \cap w\Phi^- \to \Pi$ that takes value α_i exactly k_i times and $f(\alpha) \in \operatorname{supp}\alpha$ for all $\alpha \in \Phi^+ \cap w\Phi^-$.

When a coefficient equals 0

Theorem (R. D.)

$c_{w,k_1,\ldots,k_r} > 0$ *if and only if there is a function* $f\colon \Phi^+ \cap w\Phi^- \to \Pi$ *that takes value* α_i *exactly* k_i *times and* $f(\alpha) \in \operatorname{supp}\alpha$ *for all* $\alpha \in \Phi^+ \cap w\Phi^-$.

Example ($G = SL_4$)

$\operatorname{supp}(\varepsilon_i - \varepsilon_j) = \{\alpha_i, \ldots, \alpha_{j-1}\}$

For $w = \binom{1,\ldots,r+1}{s_1,\ldots,s_{r+1}}$, $\varepsilon_i - \varepsilon_j \in \Phi^+ \cap w\Phi^-$ if and only if the numbers i and j in the bottom line form an inversion.

Let $w = \binom{1234}{2413}$

Inersions: 21, 41, 43.

$\Phi^+ \cap w\Phi^- = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_4, \varepsilon_3 - \varepsilon_4\}$

Supports: $\{\alpha_1\}$, $\{\alpha_1, \alpha_2, \alpha_3\}$, $\{\alpha_3\}$.

$c_{w,2,0,1} > 0$ since f can take values α_1, α_1, α_3.

$c_{w,2,1,0} = 0$ since no f can take value α_1 twice and α_2 once.

When a coefficient equals 1

Definition

For this talk, a *configuration* is a sequence $A, k_1, \ldots, k_r$, where $A \subseteq \Phi^+$ and $k_i \in \mathbb{Z}_{\geq 0}$.

Definition

Let $\mathcal{A} = (A, n_1, \ldots, n_r)$ be a configuration, and let $I \subseteq \Pi$.

- The *restriction* of $\mathcal{A}$ to I is the configuration $B, k_1, \ldots, k_r$, where $\alpha \in B$ iff $\alpha \in A$ and $\operatorname{supp} \alpha \cap I \neq \varnothing$, and $k_i = n_i$ for $\alpha_i \in I$, $k_i = 0$ otherwise. Notation: $(B, k_1, \ldots, k_r) = R_I(\mathcal{A})$
- The *complement* of I in $\mathcal{A}$ is the configuration $C, m_1, \ldots, m_r$, where $\alpha \in C$ iff $\alpha \in A$ and $\operatorname{supp} \alpha \cap I = \varnothing$, and $m_i = 0$ for $\alpha_i \in I$, $m_i = k_i$ otherwise. Notation: $(C, m_1, \ldots, m_r) = C_I(\mathcal{A})$

When a coefficient equals 1

Example ($G = SL_4$)

A configuration:
$\mathcal{A} = (\{\alpha_1, \alpha_1 + \alpha_2 + \alpha_3, \alpha_3\}, 2, 0, 1)$
equivalently: $\mathcal{A} = (\{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_4, \varepsilon_3 - \varepsilon_4\}, 2, 0, 1)$
Let $I = \{\alpha_1\}$.
Restriction: $R_I(\mathcal{A}) = (\{\alpha_1, \alpha_1 + \alpha_2 + \alpha_3\}, 2, 0, 0)$
Complement: $C_I(\mathcal{A}) = (\{\alpha_3\}, 0, 0, 1)$

When a coefficient equals 1

Definition

A configuration $A, k_1, \ldots, k_r$ is called a *cluster* if:

- $|A| = \sum k_i$
- There exists a function $f\colon A \to \Pi$ that takes each value $\alpha_i \in \Pi$ exactly k_i times, and $f(\alpha) \in \operatorname{supp} \alpha$ for each $\alpha \in A$.
- $\alpha \cdot \beta \geq 0$ for all $\alpha, \beta \in A$.
- For all $\alpha, \beta \in A$ such that $\alpha \cdot \beta = 0$, and for all $\alpha_i \in \operatorname{supp} \alpha \cap \operatorname{supp} \beta$ we have $k_i = 0$.

The *empty cluster* is $\varnothing, 0, \ldots, 0$.

Definition (by induction)

A configuration $\mathcal{A} = (A, k_1, \ldots, k_r)$ is called *clusterizable* if it is either a cluster or there exists a subset $I \subseteq \Pi$ such that

- $R_I(\mathcal{A})$ is a nonempty cluster, and
- $C_I(\mathcal{A})$ is clusterizable.

When a coefficient equals 1

Example ($G = SL_4$)

A configuration:
$\mathcal{A} = (\{\alpha_1, \alpha_1 + \alpha_2 + \alpha_3, \alpha_3\}, 2, 0, 1)$
equivalently: $\mathcal{A} = (\{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_4, \varepsilon_3 - \varepsilon_4\}, 2, 0, 1)$
Let $I = \{\alpha_1\}$.
Restriction: $R_I(\mathcal{A}) = (\{\alpha_1, \alpha_1 + \alpha_2 + \alpha_3\}, 2, 0, 0)$ is a cluster
Complement: $C_I(\mathcal{A}) = (\{\alpha_3\}, 0, 0, 1)$ is a cluster, therefore clusterizable
(Actually, $\mathcal{A}$ itself is also a cluster).

When a coefficient equals 1

Theorem (R. D.)

Let $w \in W$ and $k_1, \ldots, k_r \in \mathbb{Z}_{\geq 0}$.
Then $c_{w,k_1,\ldots,k_r} = 1$ if and only if $\Phi^+ \cap w\Phi^-, k_1, \ldots, k_r$ is clusterizable.

Theorem (R. D.)

The maximal degree of a multiplicity free product of Schubert divisors is:

Type of G	
A_r	$r(r+1)/2$
D_r	$r(r+1)/2 - 1$
E_r	$r(r+1)/2 - 2$, *i.e. 19, 26, 34.*

Part 3: Application: canonical dimension

Torsors of algebraic groups

Agreement

We speak about schemes over not necessarily algebraically closed fields. The schemes below don't necessarily have rational points, but they are always of finite type over the base field, separable, and reduced.

Definition

Let G be an algebraic group over a field K. A scheme E with an action $\varphi\colon G \times E \to E$ is called a *G-torsor* (or *a G-torsor over a point*) if the map $(\varphi, \mathrm{pr}_2)\colon G \times E \to E \times E$ is an isomorphism.

Example

Let a and b be two nondegenerate symmetric bilinear forms on K^n for some n. The subscheme of $\mathrm{Mat}_{n\times n}(K)$ defined by the equations (on $M \in \mathrm{Mat}_{n\times n}(K)$)

- $\det M = 1$ and
- $a(Me_i, Me_j) = b(e_i, e_j)$ for all basis vectors e_i, e_j

is a $SO(a)$-torsor.

Canonical dimension of a variety

Definition

Let X be a variety over a field K. The *canonical dimesnion of X* is

$$\mathrm{cd}(X) = \max_{\substack{L = \text{a field containing } K \\ X_L \text{ has a rational point}}} \ \min_{\substack{L_0 = \text{ a subfield of } L,\ K \subseteq L_0 \\ X_{L_0} \text{ still has a rational point}}} \operatorname{trdeg}_K L_0.$$

Example

If X has a rational point, then $\mathrm{cd}(X) = 0$.

Generally, canonical dimension measures "how far" the scheme is from having a rational point.

Canonical dimension of a group

Definition

Let G be an algebraic group over a field F. The *canonical dimension* of G understood as a group (notation: $\mathfrak{cd}(G)$) is

$$\max_{\substack{K=\text{an}\\ \text{extension of } F}} \ \max_{E=\text{a } G_K\text{-torsor}} \operatorname{cd}(E)$$

Very roughly: the *canonical dimension of a group* measures how complicated its torsors are.

Theorem (work in progress)

Let G be a simple split simply connected algebraic group over a field F. If a monomial $D_1^{k_1} \dots D_r^{k_r}$ in Schubert divisors is multiplicity-free, then $\mathfrak{cd}(G) \le \dim(G/B) - k_1 - \dots - k_r$

Corollary

Let G be as above. Then

Type of G	$\mathfrak{cd}(G) \le$
A_r	0 *(known before)*
D_r	*(3n-2)(n-1)/2 (known before)*
E_r	*17, 37, 86 (not known before)*

Canonical dimension: idea of proof

Definition

Let E be a torsor of a simple split algebraic group G over an arbitrary field, and let B be the Borel subgroup of G. The *quotient* E/B is defined as the categorical quotient $(E \times G/B)/G$ (it is known that it exists).

Theorem (kind of was known before, I finished the proof)

Let G be a simple split simply connected algebraic group over a field K, and let L be an extension of K. Let E be a G-torsor. Then the extension of scalars $\mathrm{CH}^1(E/B) \to \mathrm{CH}^1((E/B)_L)$ is an isomorphism.

Theorem (everyone uses, no reference)

Let X be a smooth scheme of finite type over a field K (conditions can be relaxed???), and let L be an extension of K. Then the extension of scalars $\mathrm{CH}(X) \to \mathrm{CH}(X_L)$ is a ring homomorphism.

Canonical dimension: idea of proof

Theorem (Karpenko, easily follows from Fulton)

Let G be a simple split simply connected algebraic group over a field K with a Borel subgroup B, and let L be an extension of K. Let E be a G-torsor. Then a class $a \in \mathrm{CH}(E/B)$ is can be written as a linear combination of irreducible subschemes with nonnegative coefficients if and only if $a_L \in \mathrm{CH}((E/B)_L)$ can be written as a linear combination of irreducible subschemes with nonnegative coefficients.

Theorem (Karpenko, easily follows from Fulton)

Let G be a simple split simply connected algebraic group over an arbitrary field with a Borel subgroup B. Let E be a G-torsor, let $X, Y \subseteq E/B$ be arbitrary subschemes.
Let $a \in \mathrm{CH}(E/B)$ (resp. $b \in \mathrm{CH}(E/B)$) be a linear combination of irreducible subschemes contained in X (resp. in Y) with nonnegative coefficients. Then $ab \in \mathrm{CH}(E/B)$ is a linear combination of irreducible subschemes contained in $X \cap Y$ with nonnegative coefficients.

Canonical dimension: idea of proof

Theorem (Karpenko, Merkurjev)

Let G, E, B be as above. Then $\mathrm{cd}(E) = \mathrm{cd}(E/B)$

Theorem (Karpenko, Merkurjev)

The canonical dimension of E/B equals the minimal dimension of a subscheme $Y \subseteq E/B$ such that there exists a rational map $E/B \dashrightarrow Y$.

Canonical dimension: idea of proof

Idea of proof of the main theorem

Take an arbitrary extension K of F. Take a G_K-torsor E. Set $L = K(E/B_K)$, then $E_L \cong G_L$ and $(E/B_K)_L \cong (G/B)_L$.
Check: $(D_i)_L$ (resp. $(Z_w)_L$) are the Schubert divisors (resp. varieties) in $G_L/B_L = (G/B)_L$, and $(D_1)_L^{n_1} \dots (D_r)_L^{n_r} (Z_w)_L = [\mathrm{pt}]$.
$\mathrm{CH}^1(E/B_K) \to \mathrm{CH}^1((G/B)_L)$ is an isomorphism, so take $C_i \in CH^1(E/B_K)$ nonnegative linear combinations of divisors mapping to $(D_i)_L$.
$C_1^{n_1} \dots C_r^{n_r}$ is a nonnegative linear combination of irreducible subschemes X_i of E/B_K of codimension $n = n_1 + \dots + n_r$.
In $(G/B)_L$, $(X_i)_L (Z_w)_L$ are nonnegative linear combination of points. But their sum is a rational point. So one product is a rational point, and the rest is 0. $(X_i)_L$ contains a rational point.
Recall: $L = K(E/B_K)$. There is a rational map $E/B_K \dashrightarrow X_i$.

Thank you for your attention!