Hardy operator on poly-trees.

P. Mozolyako
with N. Arcozzi, A. Volberg and P. Zorin-Kranich

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A plan

- The model: Hardy operator on the tree and the bi-tree
- Motivation: Hardy operator on the real line and on the plane
- Motivation: Weighted Dirichlet spaces
- Weighted Hardy operator on the poly-tree: what is known.

Hardy operator: dyadic tree

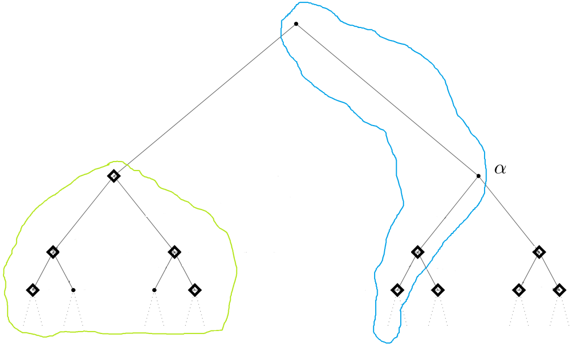
Consider a finite rooted directed dyadic tree T encoded by dyadic subintervals of $[0,1]$ ordered by inclusion. To each vertex of T we attach a non-negative number, which we call *a function, a measure, or a weight*, depending on the context.
Given a function f we define the Hardy operator and its adjoint as follows

$$\mathbf{I}f(I) := \sum_{I\subset J} f(J), \quad \mathbf{I}^* f(J) := \sum_{I\subset J} f(I).$$

We aim to study the trace inequalities

$$\sum_I ((\mathbf{I}fw)(I))^2 \mu(I) \le C^2 \sum_I f^2(I)w(I)$$
$$\sum_J ((\mathbf{I}^* g\mu)(J))^2 w(J) \le C^2 \sum_J g^2(J)\mu(J),$$

for any $g, f \ge 0$. In other words, we ask the iff conditions for the measure-weight pair (μ, w) to satisfy such that
$\mathbf{I} : L^2(T,\, dw^{-1}) \to L^2(T,\, d\mu)$ and $\mathbf{I}^* : L^2(T,\, d\mu^{-1}) \to L^2(T,\, dw)$ are bounded.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Hardy operator: dyadic tree

It turns out that now the necessary and sufficient conditions can be obtained by testing either the direct or the dual embedding.
The direct version:

$$\mu(E) \lesssim \operatorname{Cap}_w E, \quad \text{for any } E \subset T,$$

where

$$\operatorname{Cap}_w E = \inf\{\|f\|^2_{L^2(T,dw)} : \mathbf{I}fw \geq 1 \text{ on } E\}.$$

The dual version (a single-box test):

$$\sum_J \left(\sum_{I \subset J} \mu(Q \cap I)\right)^2 w(J) \lesssim \sum_{J \subset Q} \mu(J), \quad \text{for any dyadic interval } Q.$$

Moreover, we actually can weaken this condition even further:

$$\sum_{J \subset Q} (\mathbf{I}^*\mu(J))^2\, w(J) \lesssim \mathbf{I}^*\mu(Q).$$

Hardy operator on the bi-tree

Let $T^2 = T \times T$ be a bi-tree encoded by dyadic subrectangles of $[0,1)^2$. We follow 1d-case and define

$$\mathbf{I}f(Q) := \sum_{Q \subset R} f(R)$$

$$\mathbf{I}^* f(R) := \sum_{Q \subset R} f(Q),$$

and the embeddings

$$\sum_{Q} ((\mathbf{I}fw)(Q))^2 \mu(Q) \leq C^2 \sum_{Q} f^2(Q) w(Q)$$

$$\sum_{R} ((\mathbf{I}^* g\mu)(R))^2 w(R) \leq C^2 \sum_{R} g^2(R) \mu(R)$$

with (μ, w) being a (non-negative) measure-weight pair on T^2. Generally the description of the trace measure-weight pair is unknown, and we only have partial results.

Motivation

It turns out that the dyadic tree settings is a nice model for both weighted Dirichlet embeddings and Hardy operators on the line.
For the Dirichlet case we just discretize the unit disc via Whitney decomposition, and then μ on the tree is just the discrete version of μ on the disc. The weight w here depends on the parameter a, $w(I) = |I|^{a-1}$.
To handle the Hardy operator on the real line (or, which is the same, on $\mathbb{Z}_+$) we just observe that $\mathbb{Z}_+$ is a subgraph of a dyadic tree, so we can just put $w = 0$ everywhere but on one branch.

Hardy operator on the line

Let

$$I_1 f(x) := \int_0^x f(t)\,dt$$
$$I_1^* f(x) := \int_x^{+\infty} f(t)\,dt,$$

and consider the two-weighted embedding

$$\left(\int_0^\infty (I_1 f(x))^q \mu(x)\,dx\right)^{\frac{1}{q}} \leq C\left(\int_0^\infty f^p(x) v(x)\,dx\right)^{\frac{1}{p}},$$

where μ, v are non-negative weights on $\mathbb{R}_+$ and $1 < p \leq q < +\infty$. This inequality holds iff

$$\sup_{x>0} I_1^*\mu(x)^{\frac{1}{q}} I_1 w(x)^{\frac{1}{p'}} < A < +\infty,$$

where $w = v^{1-p'}$. From now on we consider only the linear case $p = q = p' = q' = 2$.

Hardy operator on $\mathbb{Z}_+$

Consider now a discrete version of the previous slide.

$$I_1 f(n) := \sum_{j=0}^{n} f(j)$$

$$I_1^* f(n) := \sum_{j=n}^{\infty} f(j),$$

the embedding now is

$$\sum_{j=0}^{\infty} (I_1 f(j))^2 \mu(j) \leq C^2 \sum_{j=0}^{\infty} f^2(j) v(j),$$

here μ, v are non-negative weights on $\mathbb{Z}_+$. The necessary and sufficient condition is as before,

$$\sup_{n>0} I_1^* \mu(n) I_1 w(n) < A^2 < +\infty.$$

Hardy operator on $\mathbb{Z}_+$

Now let us interpret $\mathbb{Z}_+$ as a directed graph, we say that $\alpha \leq \beta$ as vertices, if $\alpha \geq \beta$ as integers. In this somewhat confusing notation we have

$$I_1 f(\alpha) := \sum_{\beta \geq \alpha} f(\beta)$$

$$I_1^* f(\beta) := \sum_{\alpha \leq \beta} f(\alpha),$$

and we rewrite the embedding in a canonical way

$$\sum_{\alpha} ((I_1 f w)(\alpha))^2 \mu(\alpha) \leq C^2 \sum_{\alpha} f^2(\alpha) w(\alpha)$$

$$\sum_{\alpha} ((I_1^* g \mu)(\alpha))^2 w(\alpha) \leq C^2 \sum_{\alpha} g^2(\alpha) \mu(\alpha)$$

Hardy operator on $\mathbb{Z}^2_{++}$

As before, we consider the operators

$$I_2 f(m,n) := \sum_{j=0}^{m}\sum_{k=0}^{n} f(j,k)$$

$$I_2^* f(m,n) := \sum_{j=m}^{\infty}\sum_{k=n}^{\infty} f(j,k),$$

and the embedding

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}((I_2 fw)(m,n))^2\mu(m,n) \le C^2 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f^2(j,k)w(j,k),$$

with μ, w being non-negative weights on $\mathbb{Z}^2_{++}$. Here the situation is drastically different from 1d-case — in addition to the Muckenhoupt condition we also have to test both direct and dual embeddings on boxes:

$$\sum_{j=0}^{m}\sum_{k=0}^{n}(I_2 w)^2(j,k)\mu(j,k) \le A^2 I_2 w(m,n)$$

$$\sum_{j=m}^{\infty}\sum_{k=n}^{\infty}(I_2^*\mu)^2(j,k)w(j,k) \le A^2 I_2^*\mu(m,n), \quad \text{for any } m, n.$$

Motivation: Dirichlet space $\mathcal{D}(\mathbb{D})$

We consider spaces of analytic functions in the unit disc

$$f(z) = \sum_{n \geq 0} a_n z^n = \sum_{n \geq 0} \hat{f}(n) z^n$$

with the norm

$$\|f\|_a^2 = \sum_{n \geq 0} |\hat{f}|^2(n)(n+1)^a, \quad a \in \mathbb{R}.$$

For $a = 0$ we get the Hardy space, and $a = 1$ corresponds to the Dirichlet space,

$$\|f\|_{\mathcal{D}(\mathbb{D})}^2 = \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) + \int_{\mathbb{T}} |f(e^{it})|^2 \, \frac{dt}{2\pi},$$

where $A(\cdot)$ is the normalized surface measure on $\mathbb{D}$.

Carleson measures

Let H be a Hilbert space of continuous functions on the domain Ω. A measure μ on $\bar{\Omega}$ is called a Carleson measure, if the embedding $H \mapsto L^2(\bar{\Omega}, d\mu)$ is bounded,

$$\|f\|^2_{L^2(\bar{\Omega}, d\mu)} \lesssim \|f\|^2_H.$$

Theorem (A general one-dimensional 'theorem')

Let $f \in H_a(\mathbb{D})$, where $\|f\|^2_{H_a} = \sum_{n\geq 0} |\hat{f}|^2(n)(n+1)^a$. Then μ is Carleson for H_a if and only if

$$\mu\left(\bigcup S(I_j)\right) \lesssim \kappa_a\left(\bigcup I_j\right),$$

where $\{I_j\}$ is a finite collection of disjoint intervals on $\mathbb{T}$.

For $a = 0$ (i.e. for H^2) κ_a is the Lebesgue measure, and for $a = 1$ (Dirichlet space) κ is the logarithmic capacity.

Another description

Theorem (Local charge/energy)

Let μ be a positive measure on $\mathbb{D}$. Then μ is Carleson for the Dirichlet space on $\mathbb{D}$ iff for any dyadic interval $I \subset \mathbb{T}$ one has

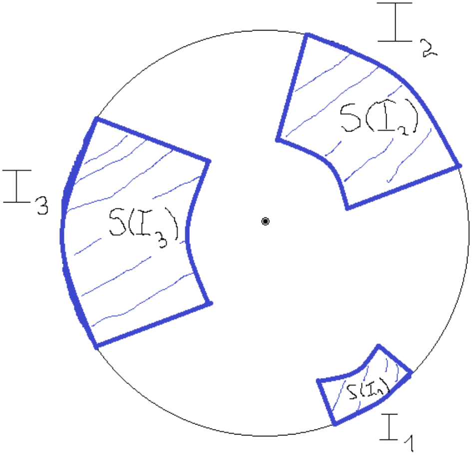
$$\sum_{J \subset I} (\mu(S(J)))^2 \lesssim \mu(S(I)).$$

In particular, if $\operatorname{supp} \mu \subset \mathbb{T}$ (there is a canonical way to push any measure to the boundary in the Dirichlet case), then we have

$$\sum_{J \subset I} (\mu(J))^2 \lesssim \mu(I).$$

Also μ is Carleson for the Hardy space iff

$$\mu(S(I)) \lesssim |I|.$$



I_2
$S(I_2)$
I_3
$S(I_3)$
$S(I_1)$
I_1

Dirichlet space $\mathcal{D}(\mathbb{D}^2)$

As before, we consider analytic functions on the bidisc $f(z,w) = \sum_{m,n\geq 0} a_{m,n} z^m w^n$. The (unweighted) Dirichlet space on $\mathbb{D}^2$ consists of analytic functions f satisfying

$$\|f\|^2_{\mathcal{D}(\mathbb{D}^2)} = \sum_{m,n\geq 0} (m+1)(n+1)|a_{m,n}|^2 < +\infty.$$

An equivalent definition is

$$\|f\|^2_{\mathcal{D}(\mathbb{D}^2)} = \int_{\mathbb{D}^2} |\partial_{zw} f(z,w)|^2 \, dA(z)\, dA(w) + \int_{\mathbb{D}} \int_{\mathbb{T}} |\partial_z f(z, e^{i\theta})|^2 \, dA(z) \, \frac{d\theta}{2\pi} + \int_{\mathbb{T}} \int_{\mathbb{D}} |\partial_w f(e^{it}, w)|^2 \, \frac{dt}{2\pi} \, dA(w) + \int_{\mathbb{T}^2} |f(e^{it}, e^{i\theta})|^2 \, \frac{dt}{2\pi} \, \frac{d\theta}{2\pi}.$$

Suggestion for the general two-dimensional theorem

Let $f \in H_{a,b}(\mathbb{D}^2)$, where $\|f\|^2_{H_{a,b}} = \sum_{m,n\geq 0} |\hat{f}|^2(m,n)(m+1)^a(n+1)^b$.
Then μ is Carleson for $H_{a,b}$ if and only if

$$\mu\left(\bigcup_{k=1}^{N} S(I_k)\times S(J_k)\right) \leq C_\mu \kappa_{a,b}\left(\bigcup_{k=1}^{N} I_k \times J_k\right),$$

where $\{I_k\}, \{J_k\}$ are finite collections of disjoint intervals on $\mathbb{T}$.
As before, for $a = b = 0$ (i.e. for $H^2(\mathbb{D}^2)$) $\kappa_{a,b}$ is the Lebesgue measure, and for $a = b = 1$ (Dirichlet space)
$\kappa_{a,b}$ is Bessel $\left(\frac{1}{2}, \frac{1}{2}\right)$- capacity (or bi-logarithmic capacity).

Dyadic maximal function

Assume that μ is a non-negative measure on $[0,1]$, and put

$$M_1 f(x) := \sup_{I \ni x} \frac{1}{\mu(I)} \int_I f \, d\mu.$$

Then $M_1 : L^2([0,1],\, d\mu) \to L^2([0,1],\, d\mu)$ is bounded.
Let us do the same for $[0,1]^2$,

$$M_2 f(x) := \sup_{R \ni x} \frac{1}{\mu(R)} \int_R f \, d\mu.$$

If μ is a product measure, the maximal operator is bounded but that's about it. Can we say something about measures μ such that $M_2 : L^2([0,1]^2,\, d\mu) \to L^2([0,1]^2,\, d\mu)$ is still a bounded operator?

Hardy operator on the bi-tree: conditions

Theorem (A. Chang, R. Fefferman)

If w is an arbitrary weight and μ is just a Lebesgue measure (that is μ is supported on the distinguished boundary $(\partial T)^2$ of the bi-tree that corresponds to $[0,1]^2$), then (μ, w) is a trace pair iff for any finite collection $\{Q_k\}$ of dyadic rectangles and $E = \bigcup Q_k$ one has

$$\sum_{Q \subset E} |Q|^2 w(Q) \lesssim |E|.$$

Theorem (N. Arcozzi, P.M., A. Volberg, P. Zorin-Kranich)

If w is a product weight, then (μ, w) is a trace pair iff for any dyadic rectangle R one has

$$\sum_{Q \subset R} (\mathbf{I}^*\mu)^2(Q) w(Q) \lesssim \mathbf{I}^*\mu(R).$$

Carleson measures on the bi-disc

Theorem (A. Chang, R. Fefferman)

A measure μ on the bi-disc is Carleson for the Hardy space $H^2(\mathbb{D}^2)$ iff for any finite collection $\{Q_k\}$ of rectangles with $E = \bigcup Q_k$ one has

$$\mu(\bigcup S(Q_k)) \lesssim |E|.$$

Theorem (N. Arcozzi, P.M., A. Volberg, P. Zorin-Kranich)

A measure μ on the bi-disc is Carleson for the Dirichlet space $\mathcal{D}(\mathbb{D}^2)$ iff for any dyadic rectangle R one has

$$\sum_{Q \subset R} (\mu(S(Q)))^2 \lesssim \mu(S(R)).$$

Hardy operator on the bi-tree: Sawyer conditions

A natural choice for the description of trace measure-weight pairs would be bi-tree version of Sawyers conditions for $\mathbb{Z}^2_{++}$:

$$\sup_Q \mathbf{I}^*\mu(Q)\mathbf{I}w(Q) < A < +\infty$$

$$\sum_{Q\subset R} (\mathbf{I}^*\mu(Q))^2 w(Q) \leq A^2 \mathbf{I}^*\mu(R), \quad \text{for any } R$$

$$\sum_{Q\subset R} (\mathbf{I}w(R))^2 \mu(R) \leq A^2 \mathbf{I}w(Q), \quad \text{for any } Q,$$

i.e. three single-box tests. This question is quite open.

Potential theory on the tree and bi-tree

From now on we assume for simplicity that $w \equiv 1$.
Given a measure μ on T or T^2 define

$$\mathbf{V}^{\mu}(Q) = (\mathbf{I}\mathbf{I}^*)(Q) = \sum_{R \supset Q} \sum_{U \subset R} \mu(U).$$

On the tree this is just a discrete version of logarithmic potential. Given a set E (a subset of T or T^2) let

$$\operatorname{Cap} E := \inf\{\mathcal{E}[\mu] : \ \mathbf{V}^{\mu}(Q) \geq 1, \ Q \in E\},$$

where

$$\mathcal{E}[\mu] = \int \mathbf{V}^{\mu} \, d\mu = \sum_{Q} (\mathbf{I}^* \mu)^2(Q)$$

is the energy of μ. By general theory there exists a unique minimizer μ_E — the equilibrium measure of E, such that $\operatorname{Cap} E = \mathcal{E}[\mu_E]$ and $V^{\mu_E} \equiv 1$ on $\operatorname{supp} \mu_E \subset E$ (we restrict ourselves to finite graphs, so no need to deal with q.a.e.).

Potential theory on the bitree: capacitary strong inequality

Now let $\mu \geq 0$, given $\lambda > 0$ consider

$$E_\lambda := \{Q : \mathbf{V}^\mu(Q) \geq \lambda\}.$$

It follows that

$$\operatorname{Cap} E_\lambda \leq \mathcal{E}\left[\frac{\mu}{\lambda}\right] = \frac{1}{\lambda^2}\mathcal{E}[\mu],$$

since $\frac{\mu}{\lambda}$ is admissible for E_λ.
Is it true that

$$\int_0^\infty \lambda \operatorname{Cap} E_\lambda \, d\lambda \leq C\mathcal{E}[\mu],$$

for some absolute constant C?
Maximum Principle:

$$sup_{Q \in \operatorname{supp} \mu} \mathbf{V}^\mu(Q) \gtrsim \sup_Q \mathbf{V}^\mu(Q),$$

then YES (Maz'ya, Adams, Hanson).

Potential theory on the bitree: capacitary strong inequality

PROBLEM: there exists $\mu \geq 0$ on T^2:

$$1 = \sup_{Q \in \operatorname{supp} \mu} \mathbf{V}^\mu(Q) < \sup_Q \mathbf{V}^\mu(Q) = \infty.$$

SOLUTION: (almost MP): if $\operatorname{supp}_{Q \in \operatorname{supp} \mu} \mathbf{V}^\mu \leq 1$ and $\lambda \geq 1$, then

$$\operatorname{Cap} E_\lambda \lesssim \frac{1}{\lambda^2 \cdot \lambda} \mathcal{E}[\mu].$$

Equivalent mixed energy estimate: let $F \subset E$, then

$$\mathcal{E}[\mu_E, \mu_F] = \int V^{\mu_E} \, d\mu_F \lesssim (\mathcal{E}[\mu_E])^{\frac{1}{2} - \frac{1}{6}} (\mathcal{E}[\mu_F])^{\frac{1}{2} + \frac{1}{6}}.$$

THANK YOU!