# Hardy–Littlewood–Sobolev inequality for $p = 1$

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Let $f$ be a function in $d$ variables. The Riesz potential $\mathrm{I}_\alpha$, $\alpha \in (0, d)$, is a convolution with the rotationally symmetric homogeneous of order $\alpha - d$ kernel:

$$\mathrm{I}_\alpha[f](x) = \int\limits_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}}\, dy. \tag{0.1}$$

It is used for several purposes. The most common is that the Riesz potential of integer order allows to represent the function in terms of its partial derivatives (there are others, say, one may measure the Hausdorff dimension of sets and measures with the help of the Riesz potential). This was first used by Sobolev to prove what is now called the Sobolev Embedding Theorem. The most important size information about the Riesz potential (also found by Sobolev) is the Hardy–Littlewood–Sobolev inequality that says $\mathrm{I}_\alpha$ is continuous as an operator between $L_p$ and $L_q$ iff $1/p - 1/q = \alpha/d$ (one may restore this condition from the homogeneity of the Riesz potential) and $1 < p < q < \infty$.

We will discuss the limit case $p = 1$, where the inequality is false (a counterexample is given by a delta-measure). This case is, in a sense, the most interesting since $L_1$ does not differ much from the space of all finite signed measures in this context. The later space is interesting from a geometric point of view. It appears that if one excludes the delta measures in a uniform way, then the endpoint inequality becomes true.