Triangular factorization of Toeplitz matrices and Wiener-Hopf operators

Roman Bessonov

SPBSU & PDMI RAS

A family $\mathcal{L}$ of subspaces in a Hilbert space H forms a chain if either $E \subset F$ or $F \subset E$ for any pair of subspaces $E, F \in \mathcal{L}$. A bounded linear operator A on H is called upper-triangular with respect to $\mathcal{L}$ if $AE \subset E$ for every $E \in \mathcal{L}$. The nest algebra $\mathcal{A}(\mathcal{L})$ consists of all bounded operators that are upper-triangular with respect to $\mathcal{L}$.

The problem of triangular factorization with respect to the chain $\mathcal{L}$ asks which bounded operators T on H admit representation

$$T = A_1^* A_2 \tag{1}$$

for some invertible operators $A_{1,2} \in \mathcal{A}(\mathcal{L})$. An operator $T \geq 0$ admitting triangular factorization (1) can always be factorized so that $A_1 = A_2$.

The famous result by D. R. Larson says that every positive bounded invertible operator admits a triangular factorization with respect to a given chain $\mathcal{L}$ in H if and only if $\mathcal{L}$ is countable.

In particular, there exists a bounded invertible operator $T \geq 0$ on $L^2[0, +\infty)$ that does not admit triangular factorization (1) with respect to the simplest continuous chain:

$$\mathcal{L}_0 = \{L^2[0, r], r > 0\}.$$

As to the author's knowledge, no concrete example of such an operator is known; the proof by D.R.Larson is highly non-constructive.

The Winer-Hopf theorem, one of the most classical theorems on integral equations, provides the triangular factorization for every positive bounded invertible Wiener-Hopf operator

$$W_\psi : f \mapsto \int_0^{+\infty} \psi(t-s) f(s)\, ds, \qquad t \geq 0,$$

with respect to the chain $\mathcal{L}_0^\perp = \{L^2[r, +\infty), r > 0\}$ (here ψ is a tempered distribution).

In 1994, L. Sakhnovich asked if every positive bounded invertible Wiener-Hopf operator admits triangular factorization with respect to $\mathcal{L}_0$.

Motivating by the theory of orthogonal polynomials and finite Toeplitz matrices, we present a construction that gives the affirmative answer to this question.