On Gaussian convex hulls

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Let T be a separable metric space. Let $X_i = \{X_i(t),\ t \in T\}$ be i.i.d. copies of a centered Gaussian process $X = \{X(t),\ t \in T\}$ with values in $\mathbb{R}^d$. Assume that X has a.s. bounded paths and consider the convex hulls

$$W_n = \mathrm{conv}\{\, X_1(t), \ldots, X_n(t),\ \ t \in T\}. \tag{1}$$

We are studying the existence of a limit shape for the sequence $\{W_n\}$.

Our work is motivated by the paper [1] inspired by an interesting implication in ecological context in estimating the home range of a herd of animals with population size n. Mathematical results of these articles consist in exact computation of a mean perimeter L_n and area A_n of W_n in the case when $d = 2$ and X is a standard Brownian motion on $T = [0, 1]$. It was shown that

$$L_n \sim 2\pi\sqrt{2\ln n}, \quad A_n \sim 2\pi \ln n, \quad n \to \infty. \tag{2}$$

The relation between L_n and A_n being the same as the relation between the perimeter and area of a circle of the radius $\sqrt{2\ln n}$, it seems credible to suppose that W_n rounds up with the growth of n. Our aim is to show that this phenomenon really occurs for all bounded Gaussian processes. One of our main results ([2], Th.1) establishes the existence with probability 1 of the limit

$$\lim_{n\to\infty} \frac{1}{\sqrt{2\ln n}} W_n = W \tag{3}$$

(in the sense of Hausdorff distance) and gives the complete description of the limit set W which is natural to call *limit shape* for convex hulls W_n. In particular case of standard Brownian motion on $[0, 1]$ the set W coincides with the unit ball $B_d(0, 1)$ of $\mathbb{R}^d$.

An interesting consequence of (3) is that the rate of the growth of the convex hulls W_n is the same for all bounded Gaussian processes.

The proof for continuous Gaussian processes may be easily deduced from the known results concerning the asymptotic of Gaussian samples (see [3]), but in the general case one needs an independent demonstration.

Let us remark in addition that if T is a singleton, $T = \{t_0\}$, and $d = 1$, then the process X is simply a real random variable and W_n is the segment $[\max\{\, X_1, \ldots, X_n,\ \}\,,\ \min\{\, X_1, \ldots, X_n,\ \}]$. It means that in some sense our study is closely connected with the classical theory of Gaussian extrema.

The report will review the above and new results on the asymptotic of W_n.

References

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