KAM perturbations with positive metric entropy

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We discuss nearly integrable systems with positive metric entropy constructed recently in [1]. To avoid technicalities, we concentrate on the following single example:

Theorem. *The standard flat metric on* $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ *admits an arbitrarily* C^∞*-small perturbation in the class of Finsler metrics, such that the geodesic flow of the perturbed metric has positive metric entropy.*

Comments and explanations:

- Finsler metrics are generalization of Riemannian metrics, the only difference is that in a Finsler manifold the norms on tangent spaces are not necessarily Euclidean. The notion of geodesic generalizes to Finsler metrics in a straightforward way.
- Positive metric entropy of the geodesic flow is equivalent to the following property: There exists a positive-measure set of geodesics that have exponentially growing Jacobi fields (in other words, such a geodesic diverges exponentially from an infinitesimally close one).

The geodesic flow of the flat torus belongs to the class of integrable Hamiltonian systems. In fact, a similar result with "Finsler metric" replaced by "Hamiltonian function" holds for every integrable system [1].

By the Liouville-Arnold theorem, the phase space of an integrable system is foliated by invariant Lagrangian tori with quasi-periodic motion on them. By Kolmogorov-Arnold-Moser (KAM) theorem, a small perturbation of an integrable system preserves invariant tori on a set of almost full measure. Hyperbolic behavior can only occur in tiny regions between these tori. This is especially interesting in 4-dimensional examples (such as geodesic flows of 2-dimensional spaces) because the invariant tori divide energy levels into narrow gaps.

References

[1] D. Burago, D. Chen, S. Ivanov, *An example of entropy non-expansive KAM-nondegenerate nearly integrable system*, preprint, `arXiv:2009.11651`.

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