Reconstruction of Riemannian manifolds from distance functions

Sergei V. Ivanov

PDMI RAS & SPBSU

The problem we consider is motivated by the following simplified micro-seismology model: Imagine a planet M with an active region U inside. Micro-earthquakes occur frequently in U. They produce spherical waves that propagate through the planet with variable sound speed. An observer can measure arrival times of waves at a dense set of points on the planet's surface F. Can one determine the structure of the active region from the data collected from many such events?

For mathematical setting, we assume that sound speed is governed by some unknown Riemannian metric on M. The data obtained from a seismic event occurred at a point $x \in U$ at time t_x, is the function T_x on F given by $T_x(y) = t_x + d(x, y)$ where $d(x, y)$ is the Riemannian arc-length distance.

The time t_x is unknown to the observer. To get rid of it, we work with the *distance difference function* D_x^F on $F \times F$, defined by

$$D_x^F(y, z) = T_x(y) - T_x(z) = d(x, y) - d(x, z), \qquad y, z \in F.$$

In general, the "observation domain" F is not always a boundary; there is another variant of the problem where F is an open set in a boundaryless manifold.

Now the problem is stated as follows: Assuming that we know the geometry of the observation domain F and the distance difference data $D(U, F) := \{D_x^F : x \in U\}$, can we determine the structure of U up to a Riemannian isometry? As any problem of this type, it includes several questions:

- **Uniqueness:** Do the data determine the unknown Riemannian structure uniquely?
- **Stability:** Can another manifold and another domain, which is very different from U, produce almost the same data as M and U? In other words, can small errors in the observed data lead to large errors in determination results?
- **Algorithmic reconstruction:** Is there an algorithm that computes (an approximation of) the unknown metric from the given data?

At the moment the uniqueness is solved in both variants (boundary and open domain) and there are partial results on stability. We will discuss the following results [1, 2]:

Theorem 1. *Let M be a complete Riemannian manifold with boundary F, and let $U \subset M$ bet an open set. Then the distance difference data $D(U, F)$ determine the topology, differential structure and metric of U uniquely up to a Riemannian isometry.*

Theorem 2. *Let M be a complete Riemannian manifold without boundary, and let $U, F \subset M$ be open sets. Then the distance difference data $D(U, F)$ determine the topology, differential structure and metric of U uniquely up to a Riemannian isometry.*

Furthermore, if $U = M$ and M is compact, the following stability property holds: Assuming a priori bounds on curvature and injectivity radius of M and on the size of F, the determination is stable with respect to the uniform topology on the space of distance difference data and $C^{1,\alpha}$ topology on the space of Riemannian manifolds.

Previously the problem was studied by several authors [3, 4] who solved it in less general cases and under additional assumptions.

References

[1] S. Ivanov, *Distance difference representations of Riemannian manifolds*, Geometriae Dedicata **207** (2020), 167–192, `arXiv:1806.05257`.

[2] S. Ivanov, *Distance difference functions on non-convex boundaries of Riemannian manifolds*, preprint, `arXiv.org:2008.13153`.

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[4] M. V. de Hoop, T. Saksala, *Inverse problem of Travel time difference functions on compact Riemannian manifold with boundary*, J. Geom. Anal. 29 (2019), no. 4, 3308–3327.