Linear operators, metric entropy and small deviation probability

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In this talk, we discuss a relation between a measure of compactness of linear operators and small ball (small deviation) probabilities of Gaussian random vectors, in particular, Gaussian processes.

Recall that compactness of a linear operator $L : \mathcal{H} \mapsto \mathcal{X}$ acting between two normed spaces is quantified by the metric entropy of the image of the unit ball. Namely, for $n \geq 0$ the diadic entropy number $e_n(L)$ is the minimal $r > 0$ such that the L-image of the unit ball of $\mathcal{H}$ may be covered in $\mathcal{X}$ by 2^n balls of radius r. Typically, $e_n(L)$ has a power decay rate, as $n \to \infty$. The study of $e_n(\cdot)$ for different classes of operators is a notorious problem of functional analysis.

Let X be a centered Gaussian random vector taking values in $\mathcal{X}$. The related small ball (small deviation) problem amounts to study the asymptotics

$$\mathbb{P}\left\{||X||_{\mathcal{X}} \leq \varepsilon\right\}, \qquad \varepsilon \to 0.$$

It has a long history in Probability and admits a number of deep applications, in particular, in Bayesian statistics and quantization theory.

There is a deep connection between these two problems, which, from the first glance, have nothing to do with each other. Namely, every centered Gaussian vector $X \in \mathcal{X}$ can be represented as the sum of a series

$$X = \sum_{j=1}^{\infty} \xi_j \, Le_j,$$

where (ξ_j) is an i.i.d. sequence of standard normal random variables, $L : \mathcal{H} \mapsto \mathcal{X}$ is an operator acting on a Hilbert space $\mathcal{H}$ and (e_j) is an orthonormal basis in $\mathcal{H}$ (its choice is irrelevant). One may say that L and X are associated. A typical result is as follows.

Let $\gamma > 1/2$. Then

$$e_n(L) \approx n^{-\gamma} \quad \text{iff} \quad \ln \mathbb{P}\left\{||X||_{\mathcal{X}} \leq \varepsilon\right\} \approx -\varepsilon^{-\frac{1}{\gamma-\frac{1}{2}}}.$$

In the talk we will discuss some examples and open problems related to this connection. For more details, see [1, Chapter 11.4].

References

[1] M.Lifshits, *Lectures on Gaussian Processes*, World Scientific, Singapore, 2014 (in English); Lan', St.Petersburg, 2016 (in Russian).