Isotropy of Tits construction

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Lie algebras of exceptional types appear in many areas of mathematics and theoretical physics. Jacques Tits in (based on earlier efforts of Hans Freudenthal) proposed a general construction of such algebras over an arbitrary field of characteristic not 2 and 3 called now the Tits construction. The inputs are a composition algebra (think of octonions for the case of E_8) and an exceptional 27-dimensional Albert algrebra (think of 3 by 3 Hermitian matrices over octonions with the operation $a \bullet b = \frac{1}{2}(ab + ba)$), and the result is a Lie algebra of type F_4, E_6, E_7 or E_8, depending on the dimension of the composition algebra. Later more symmetric version of the construction were proposed (by Ernest Vinberg, Burton and Sudbery and others); for our purposes the most convenient approach idue to Allison and Faulkner. Here the input is a so called structurable algebra with an involution (think of the tensor product of two octonion algebras) A and three constants (up to a common factor) $\gamma_1, \gamma_2, \gamma_3$. The Lie algebra is given by generators $x_{ij}(a)$ ($i, j = 1, 2, 3$, $i \neq j$, $a \in A$) subject to the relations

$$\begin{aligned}
&a \mapsto x_{ij}(a) \text{ is linear };\\
&x_{ij}(a) = x_{ji}(-\gamma_i \gamma_j^{-1} \bar{a});\\
&[x_{ij}(a), x_{jk}(b)] = x_{ik}(ab).
\end{aligned}$$

The construction is general enough to produce, say, all real forms of the exceptional Lie algebras, and the question is if the construction is surjective. Skip Garibaldi and Holger Petersson showed that it is not the case for type E_6, namely, Lie algebras of Tits index ${}^2E_6^{35}$ do not appear as a result of Tits construction. We show a similar result for type E_8, namely, that Lie algebras of Tits index E_8^{133} cannot be obtained by means of Tits construction, provided that the base field is perfect and has no odd degree extensions. The proof uses several results from different areas: a basic theory of symmetric spaces, a result of Mathieu Florence about smooth rational points of compactifications of homogeneous spaces, the Rost invariant and Chow motives.