On subgroups of linear groups over rings

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Main problem

Let $G = SL_n$ denotes the special linear with $n \geqslant 3$, and let K be a subring of a commutative ring A with 1.

Problem

Describe the lattice of subgroups lying between $G(K)$ and $G(A)$.

$$\mathscr{L} = L\big(E(K), G(A)\big)$$

$$E(R) := \langle e + re_{ij} \mid 1 \leqslant i \neq j \leqslant n,\ r \in R \rangle.$$

Standard answer

It turns out that quite often all such subgroups are closed to the groups $G(R)$ over intermediate subrings R.

Definition

The lattice $\mathscr{L}$ is called standard if it splits into disjoint union of "sandwiches"

$$L\big(E(R), N_A(R)\big),$$

where R is a subring in A, and $N_A(R)$ denotes the normalizer of $E(R)$ in $G(A)$.

For finite dimensional rings R the quotient group $N_A(R)/E(R)$ is soluble, and if Krull dimension of R is $\leqslant 1$, then it is really small and usually is computable.

Chevalley groups

One can consider G being another split classical group or even a Chevalley group. Each Chevalley group is almost uniquely defined by its root system, which is a finite set of vectors with a reach symmetry group.
Root systems of SL_n and SO_{2n} are denoted by A_{n-1} and D_n respectively. All roots in these systems has the same length. Such systems are called simply laced.
Root systems B_n and C_n for groups SO_{2n+1} and Sp_{2n} contain roots of differnt length. These systems are called doubly laced.
Furhter, $G = G(\Phi, \_)$ is a Chevalley group with root system Φ.

Doubly laced systems, 2 is invertible

Amazingly conditions for the standardness of the lattice $\mathscr{L}$ for Chevalley groups with simply and doubly laced root systems are extremly different.
Roughly speaking, for doubly laced root systems the standard description always hold, whereas for symply laced systems it requires a strong condition on the pair of rings.

Theorem (AS 2012)

Let Φ be a doubly laced root system (i. e. $\Phi = B_n, C_n, F_4$), and $1/2 \in K$. Then the lattice $\mathscr{L}$ is standard.

Doubly laced systems, 2 is not invertible

In case of multiply laced root systems, if 2 is not invertible (3 is not invertible if $\Phi = G_2$), the set of standard sandwiches must be extended.
Namely, one must add sandwiches, parameterized by admissible pairs (one additive subgroup on long roots, another on short ones).

Theorem (A. Bak, AS 2016; YaN, AS 2019)

Let $\Phi = B_n, C_n$, $n > 2$ and $2 = 0$ in K. Then the lattice $\mathscr{L}$ is standard.

Problem

Prove that the lattice $\mathscr{L}$ is standard, provided that $\Phi = F_4$ and $2 = 0$.

The following problem may be very difficult.

Problem

In the case of doubly laced root system prove of disprove that $\mathscr{L}$ is standard.

Quasi-algebraic ring extensions I

Let us define a condition on a ring extension that is conjecturally equivalent to the standardness of the lattice $\mathscr{L}$ for Chevalley groups with simply laced root system.

Definition

An element $r \in A$ is called quasi-algebraic over K, if it is annihilated by a polynomial over K with a unimodular row of coefficients.
A ring A is called quasi-algebraic over K, if all its elements are quasi-algebraic.

Roughly speaking, only integral extensions and localization of one-dimensinal rings are quasi-algebraic.

Quasi-algebraic ring extensions II

Theorem (AS 2004)

Let A be a domain and K a finitely generated algebra over a field or over $\mathbb{Z}$. A is quasi-algebraic over K iff

1. *A is an integral extension of K, or*
2. $\dim K \leqslant 1$ *and A embedds into an algebraic closure of the field of fractions of K.*

Problem

State and prove an analog of the previous theorem without additional conditions on A and K.

Nonstandard distribution of subgroups

The following theorem gives a class of counterexamples for the standardness of the lattice $\mathscr{L}$.

Theorem (AS 2010)

Let Φ be a simply laced root system, i. e. $\Phi = A_n, D_n, E_n$.
If A is not quasi-algebraic over K, then the lattice $\mathscr{L}$ is not standard.

If A is not quasi-algebraic over K, then the description of the lattice $\mathscr{L}$ includes the description of the lattice $L\big(E(F), G(F[t])\big)$, where F is a field.

Theorem (AS 2010)

Let Φ be a simply laced root system, G an adjoint Chevalley group of type Φ, and F a field.
Then there exists $g \in E(F[t])$ such that the group, generated by $E(F)$ и g, is equal to the free product $E(F) * \langle g \rangle$.

Standard distribution of subgroups for an arbitrary root system

Let $\Phi \neq A_1$ be a root system. If $\Phi = C_2 = B_2$, then assume that $1/2 \in K$.

Theorem (YaN 1983; YaN 2013)

If A is an algebraic extension of a field K, then the lattice $\mathscr{L}$ is standard.

Theorem (YaN, A. V. Yakushevich 2000)

If A is a field of fractions of a PID K, then the lattice $\mathscr{L}$ is standard.

Problem

Prove the previous theorem for a Dedekind domain K.

This is already done for $\Phi = A_n, D_n$. Thus the problem is mostly about $\Phi = E_6, E_7, E_8, G_2$.

Conjecture

The results above suggest the following conjecture.

Conjecture

Let $\Phi \neq A_1, G_2$ be a root system. If $\Phi = C_2$ assume that $1/2 \in K$.
The lattice $\mathscr{L}$ is standard if and only if one of the following conditions holds.

1. *Φ is a doubly laced root system;*
2. *A is quasi-algebraic over K.*

Idea of the proof of Nuzhin's theorem

Theorem (YaN 1983; YaN 2013)

If A is an algebraic extension of a field K, then the lattice $\mathscr{L}$ is standard.

We show the proof for $G = SL_n$.
Let R be the largest subfield of A such that $E(R) \leqslant H$.
Let $g = uhv \in H \smallsetminus N_A(R)$, where the matrices u and v are low and upper unitriangular, and h is diagonal.
$gt_{1n}(r)g^{-1} = u(ht_{1n}(r)h^{-1})u^{-1} = ut_{1n}(ar)u^{-1} \in H$ for all $r \in R$.
Multiplying g by a matrix from $E(K)$ beforehand, we may assume that $a \notin R$.
$t_{1n}(r) = ut_{1n}(r)u^{-1} \in H$ for all $r \in R$.

Dickson–Bashkirov lemma

Theorem (L. Dickson $\approx$ 1905, E. L Bashkirov 1982)

Let $R \subseteq A$ be a field extension, and let $a \in A$ be an algbraic element over R. Then $\langle t_{12}(ar), t_{21}(r) \mid r \in R \rangle = SL_2(R[a])$.

In our case the lemma implies that $ut_{1n}(a)u^{-1} = t_{1n}(a) \in H$, which is a contrudiction.

Idea of the proof for an integral field extension

In Nuzhin's proof take a root subsystem A_2 instead of A_1.
In other words, instead of $gt_{1n}(r)g^{-1}$ consider

$$gt_{1n}(r_1)t_{1\,n-1}(r_1)t_{2n}(r_3)g^{-1}, \text{ where } r_1, r_2, r_3 \in R.$$