Restriction Technique and Proof Complexity

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Suppose we have two players Optimist and Pessimist that have a family of constraints $\mathcal{H} := \{h_1, \dots, h_m\}$ on n variables $x_1, \dots, x_n \in \Omega$ where Ω is some finite space. Usually we assume that $\Omega := \{0, 1\}$ and h_i is a disjunction of some subset of variables or some small polynomial equation. Optimist believes that there is a point $a \in \Omega^n$ that satisfy all constraints and Pessimist believes in the opposite fact.

If Optimist is right then he can give a point $a \in \Omega^n$ that satisfy al constraints to Pessimist. Assuming that Pessimist may check the feasibility of a both players agreed that system $\mathcal{H}$ is satisfiable. But if the Pessimist is right we do not know any simple way to convince the Optimist that there is no solution of $\mathcal{H}$ (assuming that Optimist is a polynomial-time Turing machine). The main task of proof complexity is to quantify the size of the smallest proof required to prove that some given system $\mathcal{H}$ is unsatisfiable.

Let us focus on the main case: $\Omega := \{0, 1\}$, h_i is a disjunction and Optimist is a polynomial-time Turing machine. There are several motivations to pay attention to the sizes of proofs.

1. Without limitations of the power of Optimist any superpolynomial lower bound on the sizes of proofs will imply that **NP** $\neq$ **coNP** [CR79].

2. Even if Optimist is limited to check the application of resolution rule:

$$\frac{A \vee x \quad B \vee \neg x}{A \vee B},$$

   lower bounds on the size of proof will give us lower bounds on the running time of the popular algorithms for some **NP**-complete problems [DP60; AHI05].

3. Again if the focus on *weak* Optimist models like Resolution or so-called Nullstellensatz, lower bounds will imply lower bounds on the strong enough computational models line *monotone span programs* and *monotone circuits* [PR18; Gar+18].

In this talk we focus on the *Restriction Technique* that is the most popular technique for proving lower bounds on the size of proofs of unsatisfiability. And we discuss the limitations of this technique and give some open problems.

References

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