HARDY OPERATOR ON THE POLY-TREE

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We consider finite partially ordered sets with unique maximal element. Let Γ be a particular case of those — a poly-tree, i.e. a Cartesian product of several identical copies of finite rooted dyadic trees with order given by a product structure. A classical way to interpret this object is to encode it with *dyadic rectangles* on $\mathbb{R}^n$ (Cartesian product of usual dyadic intervals on $\mathbb{R}$) for some $n \geq 1$. The order here is a natural one and is given by inclusion. The elements of Γ are denoted by Q, R, J etc.

The Hardy operator and its 'adjoint' are

$$\mathbf{I}f(R) := \sum_{R\subset Q} f(Q)$$
$$\mathbf{I}^*f(Q) := \sum_{R\subset Q} f(R).$$

We are interested to investigate the boundedness of this operator acting from $L^2(\Gamma, w^{-1})$ to $L^2(\Gamma,\mu)$, or, which is the same, $\mathbf{I}^*$ from $L^2(\Gamma,\mu^{-1})$ to $L^2(\Gamma, w)$, where w and μ are just collections of non-negative weights attached to the elements of Γ. If, for given μ, w, the Hardy operator is bounded, we call (μ, w) *the trace measure-weight pair.*

Our main goal is to provide a characterization of such pairs. Their study can be justified due to their appearance in

- Study of (continuous) Hardy operators on $[0,+\infty)^n$, see e.g. [5];
- Carleson measure problem for analytic and harmonic weighted Dirichlet spaces on the poly-disc (a special case, the Hardy space on the bi-disc was studied in [2], see also [6] and [4]);
- Maximal operators on dyadic rectangles;
- Multiparameter Bessel potentials;
- Random walks with multiparameter time, as, for example, in [3] .

So far only the one-dimensional case, i.e. the dyadic tree situation, has been fully understood ([1] and references therein). Even the case of $n = 2$, or the bi-tree, presents considerable difficulty. Our main result is the following theorem.

Theorem 1 (N. Arcozzi, P.M., A. Volberg, P. Zorin-Kranich) *Let Γ be a bi-tree. If w is a product weight, then (μ, w) is a trace pair if and only if for any dyadic rectangle R one has*

$$\sum_{Q\subset R} (\mathbf{I}^*\mu)^2(Q)w(Q) \lesssim \mathbf{I}^*\mu(R).$$

Our hypothesis for the general weight uses Sawyer's result (again, for a specific choice of w).

Theorem 2 (Hypothesis for general weight on a bi-tree) *Let Γ be a bi-tree. The measure-weight pair (μ, w) is a trace pair if and only if they satisfy* ***three*** *single-box test conditions*

$$\sup_Q \mathbf{I}^*\mu(Q)\mathbf{I}w(Q) < A < +\infty$$
$$\sum_{Q\subset R} (\mathbf{I}^*\mu(Q))^2 w(Q) \leq A^2\mathbf{I}^*\mu(R), \quad \text{for any } R$$
$$\sum_{Q\subset R} (\mathbf{I}w(R))^2 \mu(R) \leq A^2\mathbf{I}w(Q), \quad \text{for any } Q.$$

References

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[2] Chang, Sun-Yung A. *Carleson measure on the bi-disc.* Ann. of Math. 109 (1979), no. 3, 613-620.

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[6] T. Tao, *Dyadic product H^1, BMO, and Carleson's counterexample*, preprint, pp. 1–12.