Linear operators, Metric Entropy, and Small Deviation Probabilities

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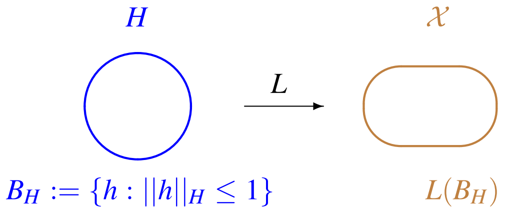

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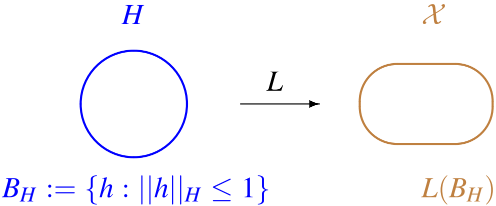

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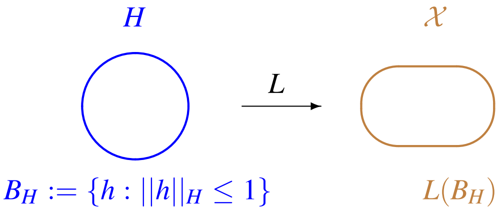

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The norm $||L||$ (half-diameter of $L(B_H)$) alone is not enough!

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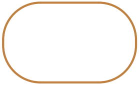

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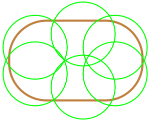

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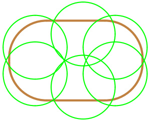

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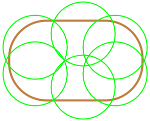

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Dyadic entropy numbers:

$$e_n(L) := \inf\{\varepsilon > 0 : N_L(\varepsilon) \le 2^n\}.$$

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For us:

The main problem in operator language:

Find the behavior of covering numbers $N_L(\varepsilon)$, as $\varepsilon \to 0$.

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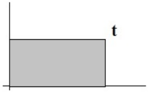

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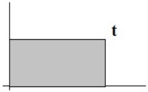

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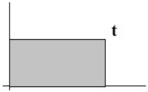

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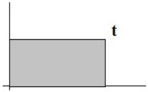

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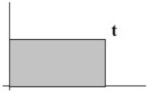

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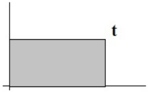

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- $c_1 n^{-1}(\ln n)^{d-1} \le e_n(L) \le c_2 n^{-1}(\ln n)^{d-1/2}$, for general d.

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A typical answer to small deviation problem is

$$\mathbb{P}\left(||X|| \le \varepsilon\right) \sim c_1 \varepsilon^a \exp\{-c_2 \varepsilon^{-b}\}, \qquad \varepsilon \to 0.$$

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The distribution of a Gaussian vector X is uniquely determined by the means and covariances of $\{g(X), g \in \mathcal{X}^*\}$.

Main example: Wiener process W – a random element in $\mathbb{C}[0,1]$ or in $L_2[0,1]$ such that $\mathbb{E}W(t) = 0$, $cov(W(s), W(t)) = \min\{s,t\}$.

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For example, let $X = W$ be a Wiener process. Then

$$\mathbb{P}\left(\sup_{0\leq t\leq 1} |W(t)| \leq \varepsilon\right) \sim \frac{4}{\pi}\ \exp\{-\frac{\pi^2}{8}\,\varepsilon^{-2}\}\,, \qquad \varepsilon \to 0,$$

and for $\mathcal{X} = L_2[0,1]$

$$\mathbb{P}\left(\int_0^1 |W(t)|^2 dt \leq \varepsilon^2\right) \sim \frac{4\varepsilon}{\sqrt{\pi}}\ \exp\{-\frac{1}{8}\,\varepsilon^{-2}\}\,, \qquad \varepsilon \to 0.$$

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If

$$\sigma_j^2 = C \left(j + \delta + O(j^{-1}) \right)^{-\alpha}, \qquad \alpha > 1, \delta \in \mathbb{R},$$

then for small deviation probabilities we have (A.I.Nazarov et al)

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Therefore, any distribution property of a Gaussian vector can be expressed in terms of operator L.

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  Let $\beta > 0, \gamma \in \mathbb{R}$. Then

  $$\ln N_L(\varepsilon) \approx |\ln \varepsilon|^{\beta} \ln |\ln \varepsilon|^{\gamma} \Leftrightarrow \varphi(r) \approx |\ln \varepsilon|^{\beta} \ln |\ln \varepsilon|^{\gamma}, \qquad \text{as } \varepsilon \to 0.$$

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- sample paths of a process X are rather smooth.
- X has good finite rank approximations

$$X \approx \sum_{j=1}^{n} \xi_j \, L(e_j), \qquad \text{as } n \to \infty.$$

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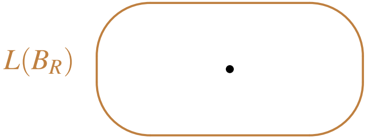

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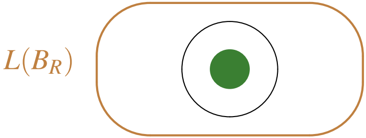

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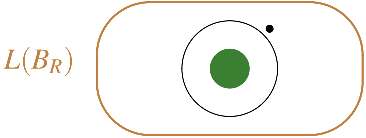

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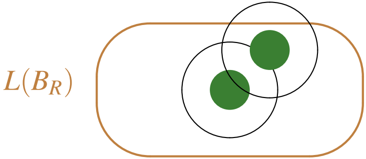

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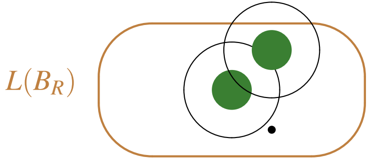

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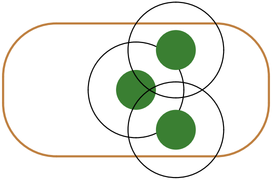

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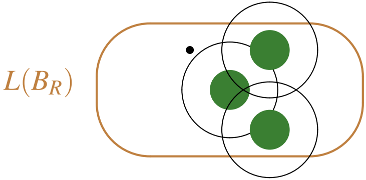

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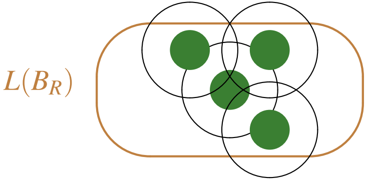

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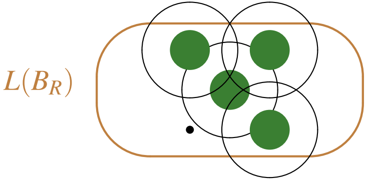

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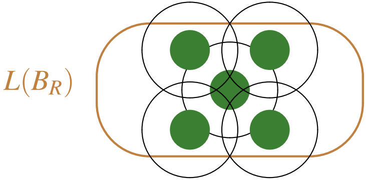

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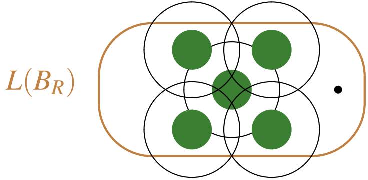

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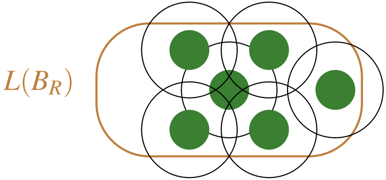

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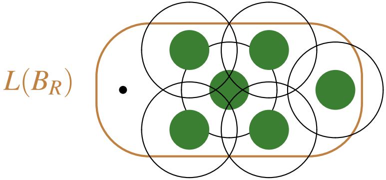

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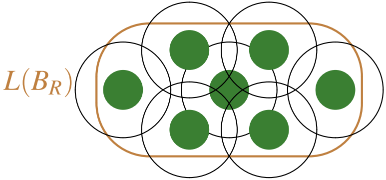

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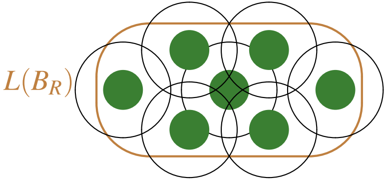

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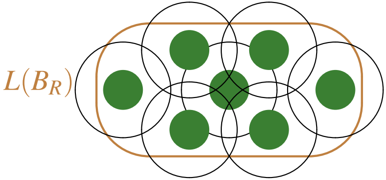

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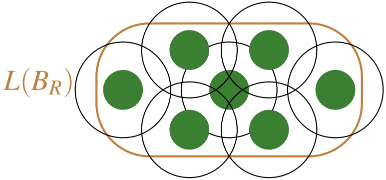

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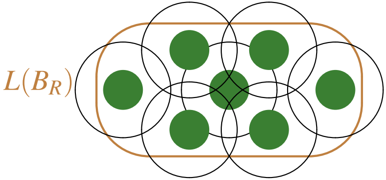

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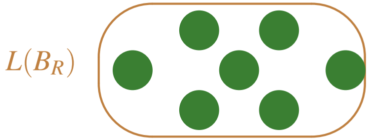

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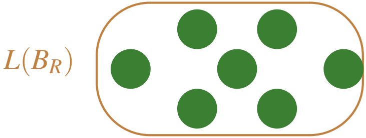

The green balls are $Lh_j + \frac{\varepsilon}{2}\, U$ where U is the unit ball in $\mathcal{X}$.

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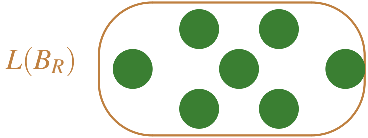

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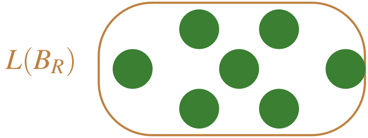

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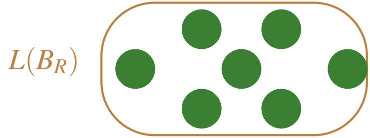

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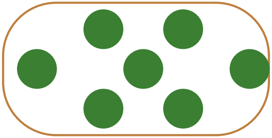

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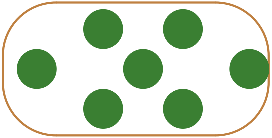

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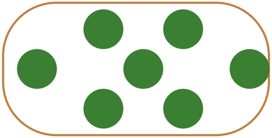

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This reads as $\mathbb{P}\big(||X|| \leq \frac{\varepsilon}{2}\big) \leq e^{R^2/2} N_L(\varepsilon/R)^{-1}$. Optimize the rhs in R !

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In the following we restrict X to the interval $[0,1]$ and evaluate the small deviations of X with respect to the uniform norm

$$||x||_\infty = \max_{t\in[0,1]} |x(t)|.$$

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Namely, consider the family of processes X_ν corresponding to absolutely continuous spectral measures

$$F_\nu(du) = \exp\{-|u|^\nu\}du, \qquad 0 < \nu < \infty.$$

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The most interesting cases are $\nu = 1$ (exponential spectrum) and $\nu = 2$ (normal spectrum).

Example: smooth Gaussian stationary processes (continued)

By using

$$\ln N_L(\varepsilon) \approx |\ln \varepsilon|^{\beta} \ln |\ln \varepsilon|^{\gamma} \Leftrightarrow \varphi(r) \approx |\ln \varepsilon|^{\beta} \ln |\ln \varepsilon|^{\gamma}, \qquad \text{as } \varepsilon \to 0.$$

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we obtain

Small deviations for exponential continuous spectra

We have, as $\varepsilon \to 0$,

$$\varphi(\varepsilon) \approx \frac{|\ln \varepsilon|^2}{\ln|\ln \varepsilon|}, \qquad 1 < \nu \le \infty,$$

and

$$\varphi(\varepsilon) \approx |\ln \varepsilon|^{1+\frac{1}{\nu}}, \qquad 0 < \nu \le 1.$$