Triangulated categories and weight structures

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Homotopy categories

$\underline{A}$=additive category; say, $\underline{A} \subset R-\mathrm{Mod}$, $\bigoplus$-closed.

Cohomological complexes: $C(\underline{A})$,
$\mathrm{Obj} = \{(M^i) \in \underline{A}^{\mathbb{Z}} + d_M^i : M^i \to M^{i+1},\ (d_M)^2 = 0\}$.

Morphisms: "$dg = gd$", that is:

$$\begin{array}{ccccccccccccc}
\dots & \xrightarrow{d_M^{-3}} & M^{-2} & \xrightarrow{d_M^{-2}} & M^{-1} & \xrightarrow{d_M^{-1}} & M^0 & \xrightarrow{d_M^0} & M^1 & \xrightarrow{d_M^1} & M^2 & \xrightarrow{d_M^2} & \dots \\
 & & \downarrow g^{-2} & & \downarrow g^{-1} & & \downarrow g^0 & & \downarrow g^1 & & \downarrow g^2 & & \\
\dots & \xrightarrow{d_N^{-3}} & N^{-2} & \xrightarrow{d_N^{-2}} & N^{-1} & \xrightarrow{d_N^{-1}} & N^0 & \xrightarrow{d_N^0} & N^1 & \xrightarrow{d_N^1} & N^2 & \xrightarrow{d_N^2} & \dots
\end{array}$$

Homotopy category $K(\underline{A})$: $g_1 \backsim g_2 \iff$
$g_1^i - g_2^i = d_N^{i-1} \circ h^i + h^{i+1} \circ d_M^i$ for some $h^i : M^j \to N^{j-1}$.

(Bounded versions: 0 terms in degrees $\gg, \ll 0$.)

Triangulated.

1. Shift: $M[1]^i = M^{i+1}$.

2. *Distinguished triangles*: $T = (\dots \to A \xrightarrow{f} B \to C \to A[1] \to \dots)$ that gives long exact sequences of (co)homology. f determines T.

Cohomology of complexes: $H^i(M) = \mathrm{Ker}(d_M^i)/\,\mathrm{Im}(d_M^{i-1})$.

Derived categories

Why no "nice functors" (from Top) into $C(R-\mathrm{Mod})/K(R-\mathrm{Mod})$? If $C(X)$ depends on a triangulation/cover, we only have

$$C(X) \xleftarrow{\;f\;} C''(X) \xrightarrow{\;f'\;} C'(X)$$

Yet all $H^i(f)$ are bijective.
Localization: invert all these f in $K(R-\mathrm{Mod})$ to obtain a triangulated $D(R)$ with morphisms of the form gf^{-1}. "More (complicated) than" just $R{-}\mathrm{Mod}^{\mathbb{Z}}$.

Canonical truncations & t-structures; "t-pieces".

$\exists$ endofunctors on $D(R)$ that kill cohomology in degrees $\geq i$ and $\leq i$ + distinguished triangles
$(L_tC =)t^{\leq 0}C \to C \to t^{\geq 1}C(= R_tC) \to t^{\leq 0}C[1]$.

[BBD82]: on a triangulated $\underline{C}$. Also, $t^{\leq 0} \perp t^{\geq 1}$.

Composing t-truncations one obtains cohomology $\underline{C} \to \underline{Ht} = t^{\leq 0} \cap t^{\geq 0}$ (example: perverse sheaves).

Stupid truncations and weight structures

For a complex $M = (M^i)$, stupid truncations:

$$\begin{array}{ccccccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M^0 & \longrightarrow & M^1 & \longrightarrow & M^2 & \longrightarrow & \dots \\
 & & & & & & \downarrow & & & & & & \\
\dots & \longrightarrow & M^{-2} & \longrightarrow & M^{-1} & \longrightarrow & M^0 & \longrightarrow & M^1 & \longrightarrow & M^2 & \longrightarrow & \dots \\
 & & & & & & \downarrow & & & & & & \\
\dots & \longrightarrow & M^{-2} & \longrightarrow & M^{-1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

Not canonical (for $0 \cong (\dots \to 0 \to A \to A \to \dots)$)!

In a triangulated $\underline{C}$, $\underline{A} \subset \operatorname{Obj} \underline{C}$ is *retraction-closed* if $\underline{A} = \operatorname{Kar}_{\underline{C}}(\underline{A})$ ={direct $\underline{C}$-summands of elements}.

Definition 1. [Weight structures: [Pau08],[Bon10].]

(i) $\underline{C}_{w\le 0}, \underline{C}_{w\ge 0} \subset \underline{C}$ are retraction-closed.

(ii) $\underline{C}_{w\le 0} \subset \underline{C}_{w\le 0}[1]$; $\underline{C}_{w\ge 0}[1] \subset \underline{C}_{w\ge 0}$.

(iii) **Orthogonality**. $\underline{C}_{w\le 0} \perp \underline{C}_{w\ge 0}[1]$.

(iv) **Weight decompositions**. $\forall M \in \operatorname{Obj} \underline{C}$

$$\exists\ L_wM \to M \to R_wM \to L_wM[1]:$$

$L_wM \in \underline{C}_{w\le 0}$ and $R_wM \in \underline{C}_{w\ge 0}[1]$.

The *heart*: $\underline{Hw} = \underline{C}_{w\ge 0} \cap \underline{C}_{w\le 0} \subset \underline{C}$; non-canonical "$w$-pieces". Also, any additive $F : \underline{Hw} \to R-\text{Mod}$

extends to (w-pure) homology $\underline{C} \to R-\mathrm{Mod}$; $[-]_{Euler}$.

Theorem 2. 1. $w^{op} = (\underline{C}_{w\geq 0}, \underline{C}_{w\leq 0})$ is a weight structure on $\underline{C}^{op}$ (self-duality).

2. $\forall i > 0$, $\underline{Hw} \perp \underline{Hw}[i]$ (*connective*).

3. "Usually/almost" $\exists$ ([Sos19],[Bon18]) an exact WC : $\underline{C} \to K(\underline{Hw})$; it kills *$w$-degenerate objects* only.

Examples and construction

Theorem 3. $\underline{A} \subset \underline{C}$ is connective & additive, "(Kar)-generates $\underline{C}$" $\Rightarrow \exists! \; w$: $\underline{A} \subset \underline{Hw} = \mathrm{Kar}_{\underline{C}}\,\underline{A}$; bounded: $\mathrm{Obj}\,\underline{C} = \cup_{i\in\mathbb{Z}}\underline{C}_{w\leq 0}[i] = \cup_{i\in\mathbb{Z}}\underline{C}_{w\geq 0}[i]$.

Examples: Projective($=\mathrm{Kar}(\{\bigoplus_I R\})$) modules $\subset D(R) \to K(\mathrm{Free}\,R-\mathrm{Mod})$;*injective* modules $+\; w_{Inj}$;

Various Voevodsky motives ("universal" cohomology from varieties into triangulated DM); conservative $t_{DM} : DM^{gm} \to K^b(\mathrm{Chow})$ (improves [GiS96]);
Coproducts of $S^0 \in \mathrm{Obj}\,SH$ give cellular towers ($\underline{Hw} \cong \mathrm{Free\,Ab}$) + singular homology; equivariant spheres in $SH(G)$ $\forall$ compact Lie G (Bredon homology);
Hodge-theoretic examples.

Weight structures can be *glued*; "often" induce weight structures on localizations.

Weight filtrations on (co)homology

Morphisms of complexes extend to stupid decompositions.

An easy generalization: $\forall g : M \to N$
$\exists\ L_w g : L_w M \to L_w N$ "compatible with" g.
Thus $\forall H : \underline{C} \to R-\mathrm{Mod}, \mathrm{Sets}$ we have:

$$\begin{array}{ccc} H(L_w M) & \longrightarrow & H(M) \\ \Big\downarrow{\scriptstyle H(L_w g)} & & \Big\downarrow{\scriptstyle g} \\ H(L_w N) & \longrightarrow & H(N) \end{array}$$

$\implies \mathrm{Im}(H(L_w M) \to H(M))$ is $\underline{C}$-functorial.

Vast generalization of Deligne's weight filtrations; also of his spectral sequences. Also, improved functoriality (say, for functors on motives).

Classically, to obtain this data for smooth U/k one needs $X \supset U$, X is proper ("compact") + smooth+ "nice $X \setminus U$". To compare two X's one needs a third one + degeneration of spectral sequences for H.

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