A New Life of the Old Sieve

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First guess, then prove.
George Pólya

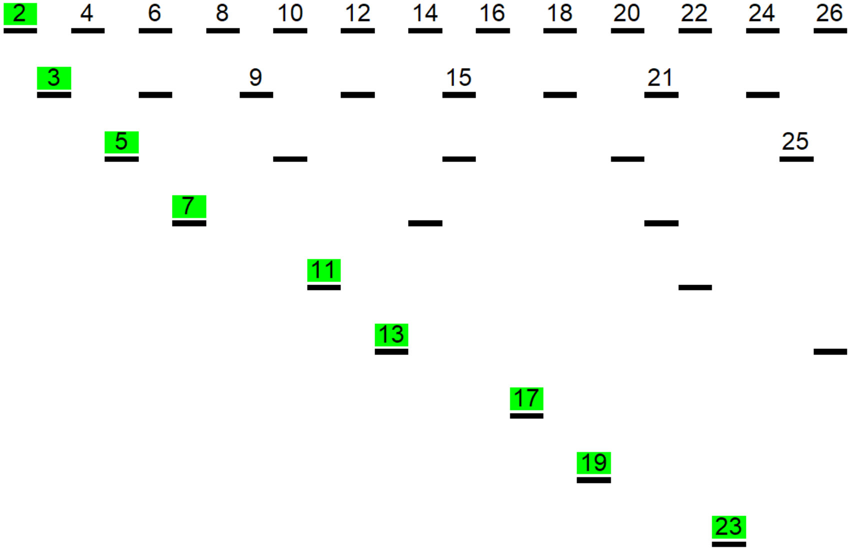
Plan of the talk

1. Sieve of Eratosthenes
2. Riemann's zeta function
3. Approximations of infinite Dirichlet series by finite Dirichlet series
4. Expected number-theoretical meanings of the coefficients of a particular approximation

Part 1

Sieve of Eratosthenes

Sieve of Eratosthenes (276–194 B. C.)



Part 2

Riemann's zeta function

Georg Friedrich Bernhard Riemann (1826–1866)

Riemann's zeta function:

$$\zeta(s) = 1^{-s} + 2^{-s} + \ldots + n^{-s} + \ldots$$

The series converges in the half-plane $\mathrm{Re}(s) > 1$ and defines a function that can be analytically extended to the entire complex plane except for the point $s = 1$, its only (and simple) pole.

Leonhard Euler (1707–1783)

$$\zeta(s) = 1^{-s} + 2^{-s} + \ldots + n^{-s} + \ldots$$

Alternating zeta function:

$$\begin{aligned}\eta(s) &= \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s} \\ &= (1 - 2 \times 2^{-s})\zeta(s)\end{aligned}$$

The series converges in the half-plane $\mathrm{Re}(s) > 0$ and defines an entire function

$$0 = \zeta(-2) = \cdots = \zeta(-2m) = \ldots$$

$-2, -4, \ldots$ are called *trivial zeroes*

Euler identity $\equiv$ The Fundamental Theorem of Arithmetic

Theorem (L. Euler [1737])

$$1^{-s} + 2^{-s} + \cdots + n^{-s} + \ldots \quad = \quad \prod_{p \text{ is prime}} \frac{1}{1 - p^{-s}}$$

Proof.

$$\begin{aligned}\prod_{p \text{ is prime}} \frac{1}{1 - p^{-s}} \quad &= \quad \prod_{p \text{ is prime}} \left(1 + p^{-s} + p^{-2s} + p^{-3s} + \ldots\right) \\ &= \quad 1^{-s} + 2^{-s} + 3^{-s} + \cdots + n^{-s} + \ldots\end{aligned}$$

The infinitude of prime numbers

Euler identity

$$1^{-s} + 2^{-s} + \cdots + n^{-s} + \ldots \quad = \quad \prod_{p \text{ is prime}} \frac{1}{1 - p^{-s}}$$

Theorem (Euclid). *There are infinitely many prime numbers.*

New proof (Euler). If the number of primes would be finite, then the (divergent) harmonic series would have finite value:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots = \prod_{p \text{ is prime}} \frac{1}{1 - \frac{1}{p}}$$

Distribution of Prime Numbers

$$\pi(x) = \text{the number of primes not exceeding } x$$

Carl Friedrich Gauss (numerical observation made at the age 15 or 16)

$$\pi(x) \approx \mathrm{Li}(x) = \int_2^x \frac{1}{\ln(t)}\mathrm{d}t \approx \frac{x}{\ln(x)}$$

Theorem (Riemann [1859])

$$\pi(x) = \mathrm{Li}(x) - \frac{1}{2}\mathrm{Li}(x^{\frac{1}{2}}) + \sum_{\zeta(\rho)=0} \mathrm{Li}(x^{\rho}) + \text{smaller terms}$$

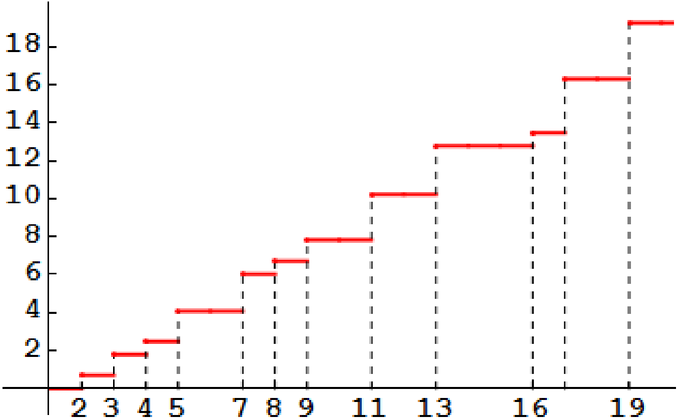
Pafnuty Lvovich Chebyshev (1821–1894)

$$\pi(x) = \sum_{\substack{p \leq x \\ p \text{ is a prime}}} 1$$

$$\begin{aligned} \psi(x) &= \sum_{\substack{q \leq x \\ q \text{ is a power} \\ \text{of a prime } p}} \ln(p) \\ &= \ln(\mathrm{LCM}(1, 2, \ldots, \lfloor x \rfloor)) \\ &\approx \ln(x)\pi(x) \\ &\approx x \end{aligned}$$

Chebyshev's function $\psi(x)$

$$\psi(x) = \sum_{\substack{q \leq x \\ q \text{ is a power} \\ \text{of a prime } p}} \ln(p) = \ln(\mathrm{LCM}(1,2,\ldots,\lfloor x \rfloor))$$



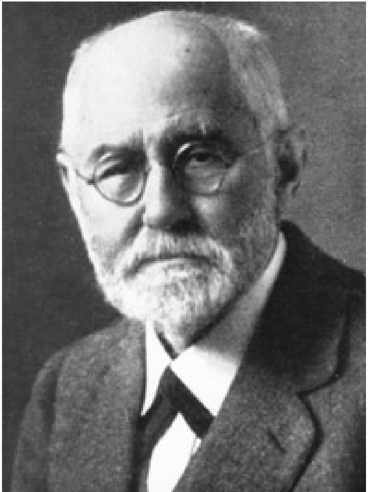
Hans Carl Friedrich von Mangoldt (1854–1925)

Theorem (Riemann [1859])

$$\pi(x) = \mathrm{Li}(x) - \frac{1}{2}\mathrm{Li}(x^{\frac{1}{2}}) + \sum_{\zeta(\rho)=0} \mathrm{Li}(x^{\rho}) + \text{smaller terms}$$

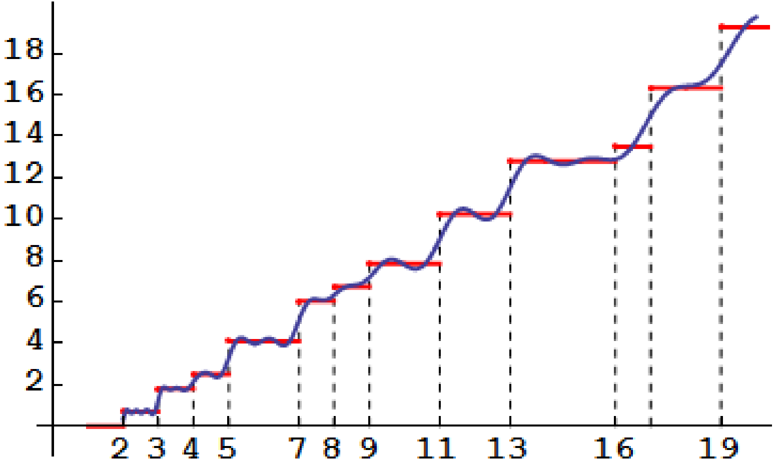
Theorem (von Mangoldt [1895])

$$\psi(x) = x - \sum_{\zeta(\rho)=0} \frac{x^{\rho}}{\rho} - \ln(2\pi)$$



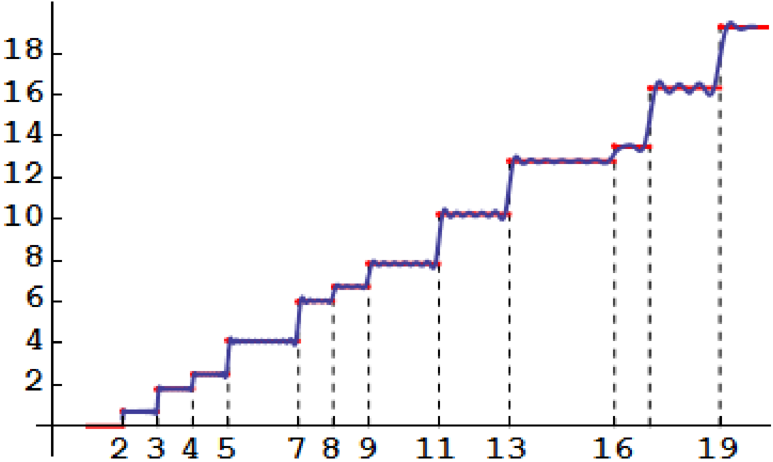
Theorem of Hans Carl Friedrich von Mangoldt

$$\psi(x) \approx x - \sum_{\substack{\zeta(\rho)=0 \\ |\rho|<50}} \frac{x^\rho}{\rho} - \ln(2\pi)$$



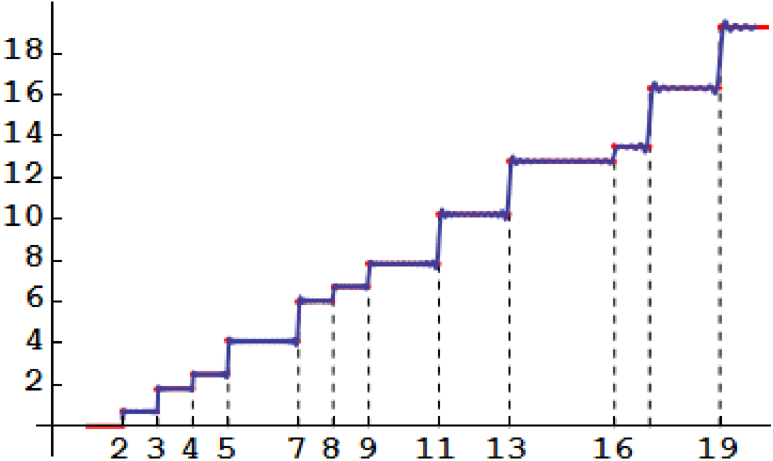
Theorem of Hans Carl Friedrich von Mangoldt

$$\psi(x) \approx x - \sum_{\substack{\zeta(\rho)=0 \\ |\rho|<200}} \frac{x^\rho}{\rho} - \ln(2\pi)$$



Theorem of Hans Carl Friedrich von Mangoldt

$$\psi(x) \approx x - \sum_{\substack{\zeta(\rho)=0 \\ |\rho|<400}} \frac{x^\rho}{\rho} - \ln(2\pi)$$



Euler and Hadamard product ⇒ Theorem of von Mangoldt

$$\begin{aligned}\zeta(s) &= \prod_{p \text{ is prime}} \frac{1}{1-p^{-s}} \\ &= \pi^{\frac{s}{2}} \frac{\prod_{\rho}\left(1-\frac{s}{\rho}\right)}{2(s-1)\Gamma\left(1+\frac{s}{2}\right)}\end{aligned}$$

$$\psi(x) = \sum_{\substack{q \leq x \\ q \text{ is a power} \\ \text{of a prime } p}} \ln(p) = x - \sum_{\zeta(\rho)=0} \frac{x^{\rho}}{\rho} - \ln(2\pi)$$

Part 3

Approximations of infinite Dirichlet series by finite Dirichlet series

Finite Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad \zeta_N(s) = \sum_{n=1}^{N} \alpha_{N,n} n^{-s}$$

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s} \qquad \eta_N(s) = \sum_{n=1}^{N} \delta_{N,n} n^{-s}$$

Numerical examples

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s} \qquad \eta(1) = \ln(2) = 0.693147180...$$

$$\eta_N(s) = \sum_{n=1}^{N} (-1)^{n+1} n^{-s} \qquad \eta_{1000}(1) = 0.69264...$$

$$\eta_N(s) = \sum_{n=1}^{N-1} (-1)^{n+1} n^{-s} + \frac{1}{2}(-1)^{N+1} N^{-s} \qquad \eta_{1000}(1) = 0.693147430...$$

Approximation proposed by Peter Borwein (1953 – 2020)

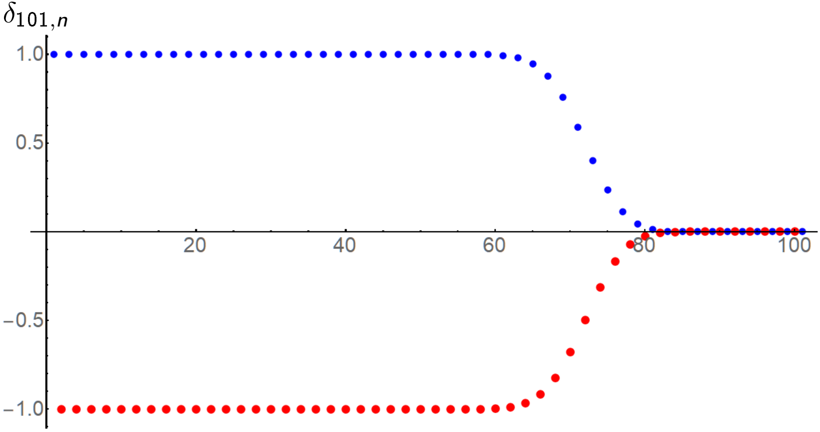
$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^s$$

$$\delta_{N,n} = (-1)^{n+1} \left(1 - \frac{\beta_{N,n}}{\beta_{N,N+1}}\right) \qquad \beta_{N,n} = N \sum_{i=1}^{n} \frac{4^{i-1}(N+i-2)!}{(N-i+1)!(2i-2)!}$$

$$\eta_N(s) = \sum_{n=1}^{N} \delta_{N,n} n^{-s}$$

$$\begin{aligned} \eta_{30}(1) &= 0.693147180559945311 25\ldots \\ \eta(1) = \ln(2) &= 0.693147180559945309417\ldots \end{aligned}$$

Borwein's coefficients $\delta_{101,n}$, red for even n, blue for odd n



$$\delta_{N,n} = (-1)^{n+1}\left(1 - \frac{\beta_{N,n}}{\beta_{N,N+1}}\right) \qquad \beta_{N,n} = N\sum_{i=1}^{n} \frac{4^{i-1}(N+i-2)!}{(N-i+1)!(2i-2)!}$$

The main "message" of the present talk

Certain finite approximations to the (alternating) zeta function can have their *own* very interesting properties

Here "own" means that such properties cannot be stated in terms of the zeta function itself

Borwein's approximation (repeated)

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^s$$

$$\delta_{N,n} = (-1)^{n+1} \left(1 - \frac{\beta_{N,n}}{\beta_{N,N+1}}\right) \qquad \beta_{N,n} = N \sum_{i=1}^{n} \frac{4^{i-1}(N+i-2)!}{(N-i+1)!(2i-2)!}$$

$$\eta_N(s) = \sum_{n=1}^{N} \delta_{N,n} n^{-s}$$

This definition is *syntactical*

Our semantic definition of $\eta_N(s)$

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^s \qquad (*)$$

$$\eta_N(s) = \sum_{n=1}^{N} \delta_{N,n} n^{-s} \qquad (**)$$

Let us define numbers $\delta_{N,n}$ by the following conditions:

- the finite sum $(**)$ has $N-1$ common zeros with the infinite sum $(*)$
- $\delta_{N,1} = 1$

Zeros of $\zeta(s)$ and $\eta(s)$

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad \eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s} = (1 - 2 \times 2^{-s})\zeta(s)$$

L. Euler: the *trivial zeros* $\quad 0 = \zeta(-2) = \cdots = \zeta(-2m) = \ldots$

The non-trivial zeros come in conjugate pairs:

$$\cdots = \zeta(\overline{\rho_3}) = \zeta(\overline{\rho_2}) = \zeta(\overline{\rho_1}) = 0 = \zeta(\rho_1) = \zeta(\rho_2) = \zeta(\rho_3) = \ldots$$

The Riemann Hypothesis: $\rho_n = \frac{1}{2} + \mathrm{i}\gamma_n \qquad 0 < \gamma_1 < \gamma_2 < \gamma_3 < \ldots$

Zeros of $(1 - 2 \times 2^{-s})$: $\;s_k = 1 + \dfrac{2\pi k}{\ln(2)}\mathrm{i}, \quad k = 0, \pm 1, \pm 2, \ldots$

The zero for $k = 0$ is cancelled by the pole of $\zeta(s)$ at $s = 1$

Formal definition of $\eta_N(s)$

$$\eta_N(s) = 1 + \sum_{n=2}^{N} \delta_{N,n} n^{-s}$$

Let $N = 2M + 1$ and let η_N be defined by the condition

$$\eta_N(\overline{\rho_M}) = \cdots = \eta_N(\overline{\rho_1}) = 0 = \eta_N(\rho_1) = \cdots = \eta_N(\rho_M)$$

More explicit definition of $\eta_N(s)$

$$N = 2M + 1$$

$$\tilde{\eta}_N(s) = \begin{vmatrix} 1 & 1 & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ n^{-\overline{\rho_1}} & n^{-\rho_1} & \dots & n^{-\overline{\rho_M}} & n^{-\rho_M} & n^{-s} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ N^{-\overline{\rho_1}} & N^{-\rho_1} & \dots & N^{-\overline{\rho_M}} & N^{-\rho_M} & N^{-s} \end{vmatrix} = \sum_{n=1}^{N} \tilde{\delta}_{N,n} n^{-s}$$

$$\eta_N(s) = 1 + \sum_{n=2}^{N} \delta_{N,n} n^{-s} = \frac{1}{\tilde{\delta}_{N,1}} \tilde{\eta}_N(s)$$

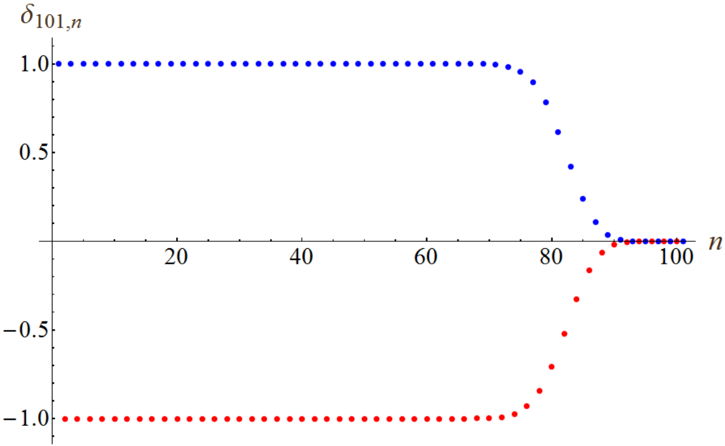
$$\delta_{N,n} = \frac{\tilde{\delta}_{N,n}}{\tilde{\delta}_{N,1}}$$

Explicit definition of $\delta_{N,n}$

$$\tilde{\delta}_{N,n} = (-1)^{n+1} \begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (n-1)^{-\overline{\rho_1}} & (n-1)^{-\rho_1} & \dots & (n-1)^{-\overline{\rho_M}} & (n-1)^{-\rho_M} \\ (n+1)^{-\overline{\rho_1}} & (n+1)^{-\rho_1} & \dots & (n+1)^{-\overline{\rho_M}} & (n+1)^{-\rho_M} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ N^{-\overline{\rho_1}} & N^{-\rho_1} & \dots & N^{-\overline{\rho_M}} & N^{-\rho_M} \end{vmatrix}$$

$$\delta_{N,n} = \frac{\tilde{\delta}_{N,n}}{\tilde{\delta}_{N,1}}$$

Coefficients $\delta_{101,n}$, red for even n, blue for odd n



The non-trivial zeros of zeta function are as clever as Borwein!

Formal definition of $\eta_N(s)$ (repeated)

$$\eta_N(s) = 1 + \sum_{n=2}^{N} \delta_{N,n} n^{-s}$$

Let $N = 2M + 1$ and let η_N be defined by the condition

$$\eta_N(\overline{\rho_M}) = \cdots = \eta_N(\overline{\rho_1}) = 0 = \eta_N(\rho_1) = \cdots = \eta_N(\rho_M)$$

This definition doesn't distinguish $\zeta(s)$ and $\eta(s) = (1 - 2 \times 2^{-s})\zeta(s)$

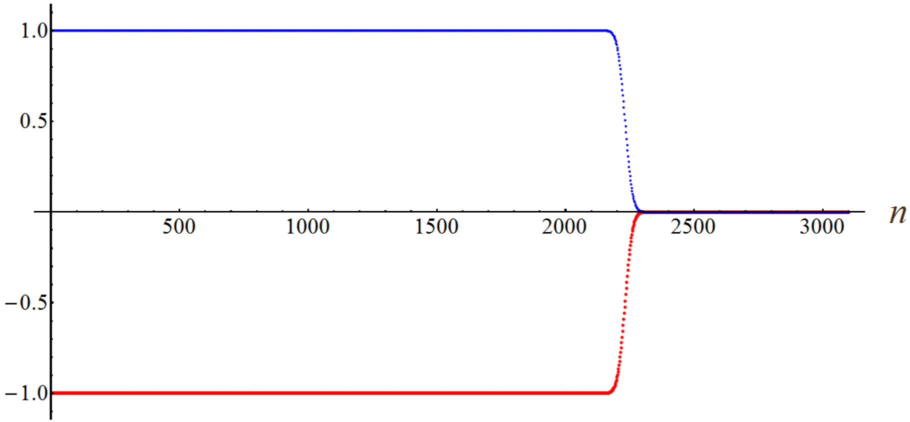
$$\cdots = \zeta(\overline{\rho_3}) = \zeta(\overline{\rho_2}) = \zeta(\overline{\rho_1}) = 0 = \zeta(\rho_1) = \zeta(\rho_2) = \zeta(\rho_3) = \dots$$
$$\cdots = \eta(\overline{\rho_3}) = \eta(\overline{\rho_2}) = \eta(\overline{\rho_1}) = 0 = \eta(\rho_1) = \eta(\rho_2) = \eta(\rho_3) = \dots$$

The non-trivial zeros of zeta function are as clever as Euler!

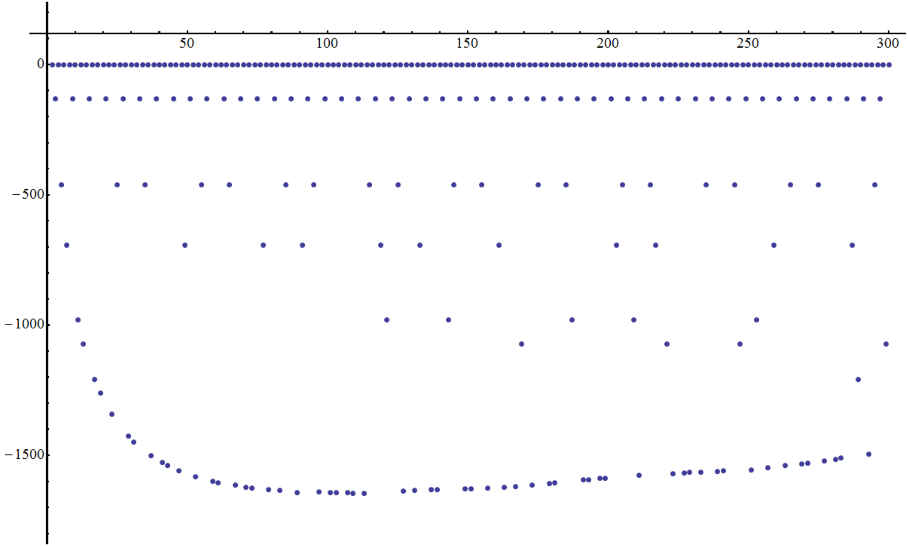
Part 4

Some observations on number-theoretical meanings of the coefficients of our finite Dirichlet series

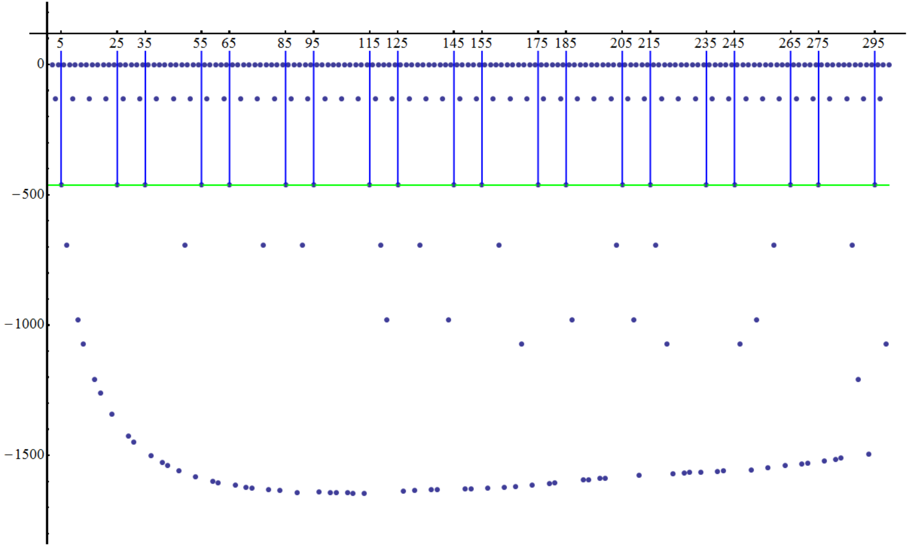
Coefficients $\delta_{3101,n}$, red for even n, blue for odd n



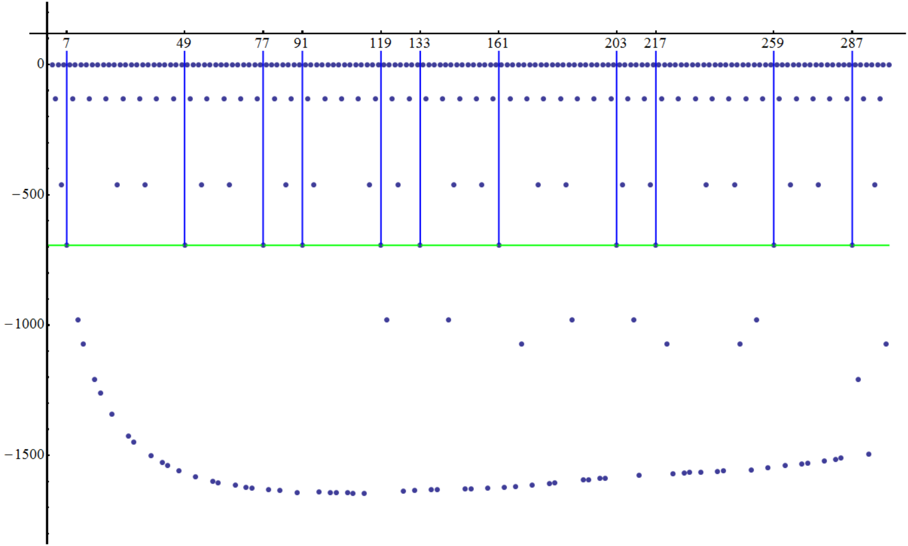
Plot of $\log_{10} |\delta_{3101,n} - 1|$



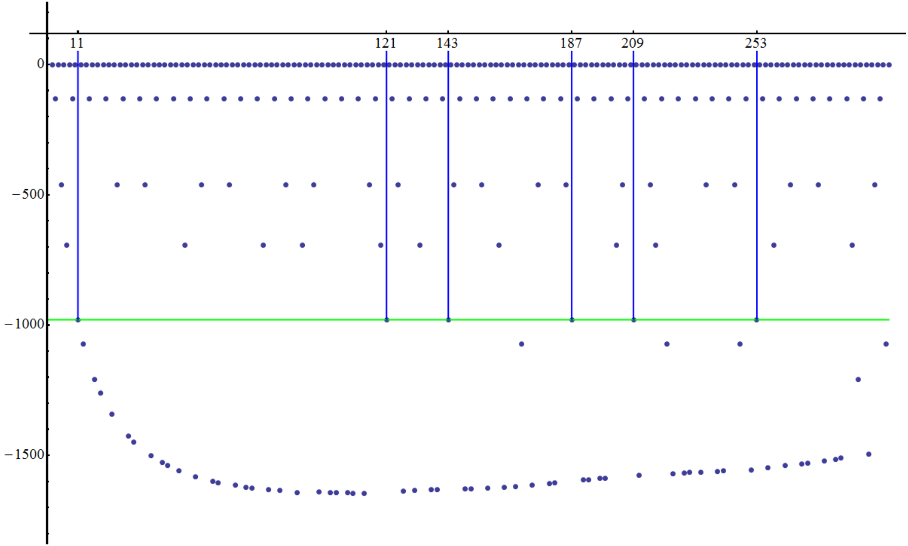
Plot of $\log_{10} |\delta_{3101,n} - 1|$



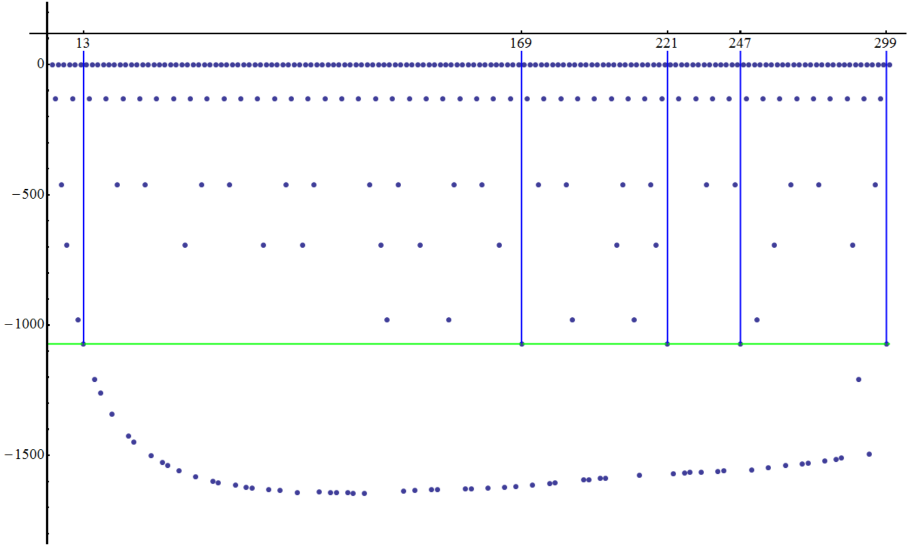
Plot of $\log_{10} |\delta_{3101,n} - 1|$



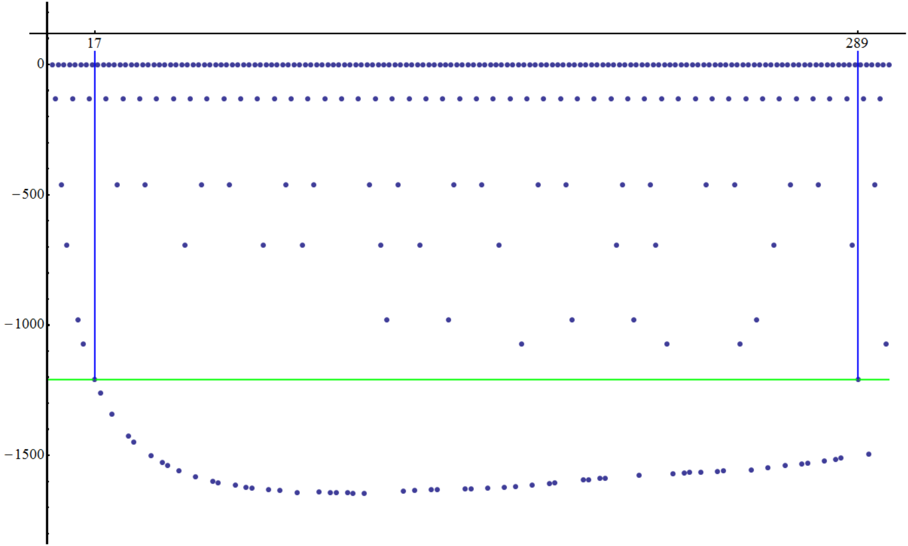
Plot of $\log_{10} |\delta_{3101,n} - 1|$



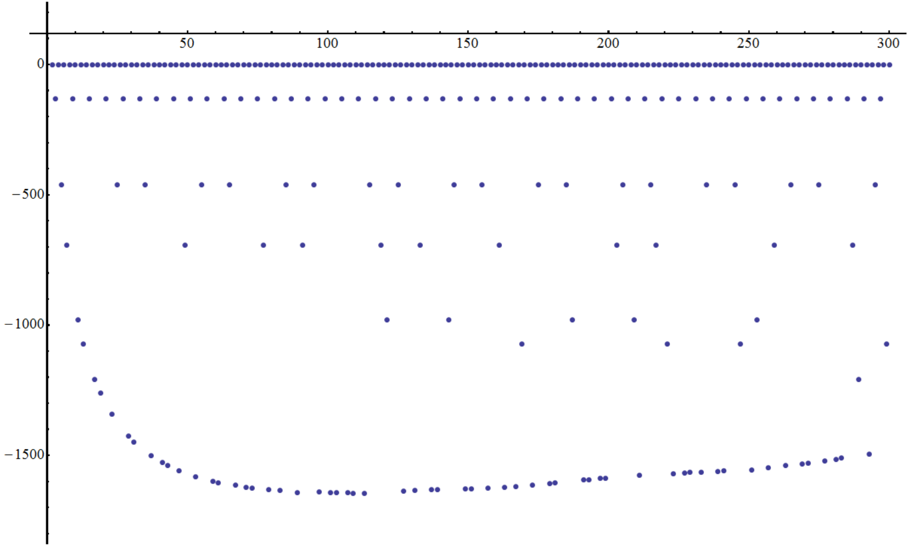
Plot of $\log_{10} |\delta_{3101,n} - 1|$



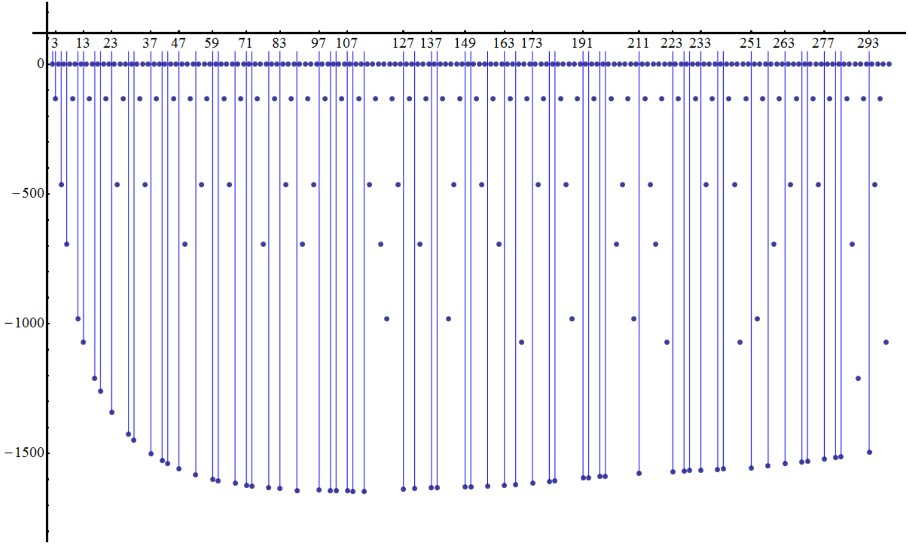
Plot of $\log_{10} |\delta_{3101,n} - 1|$



Plot of $\log_{10} |\delta_{3101,n} - 1|$

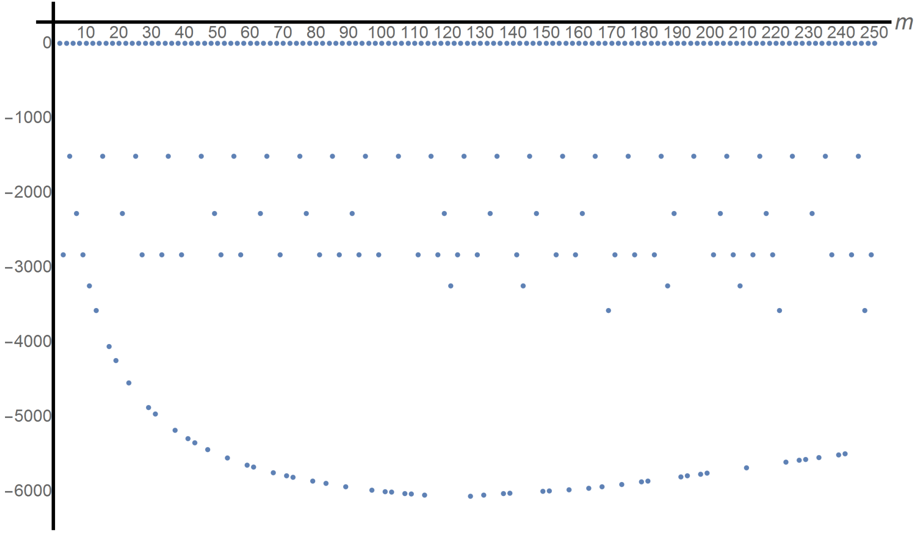


Plot of $\log_{10} |\delta_{3101,n} - 1|$ = Sieve of Eratosthenes

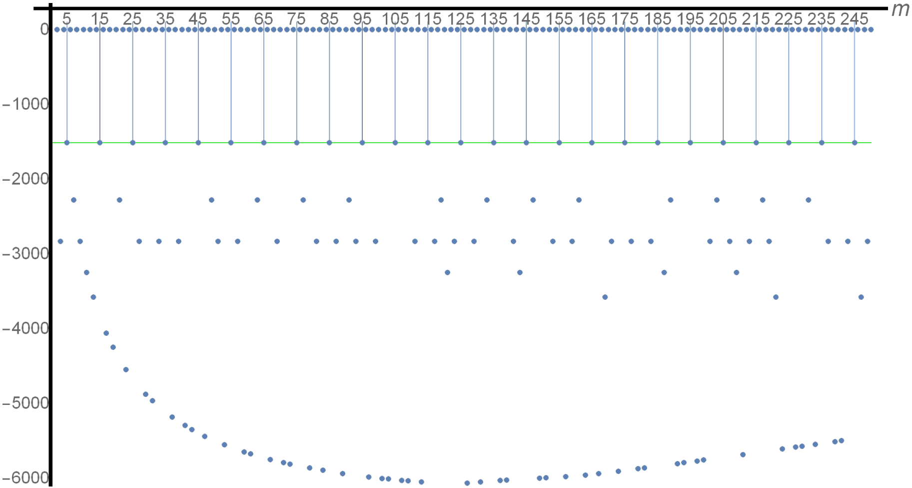


The zeros of zeta function are as clever as Eratosthenes!

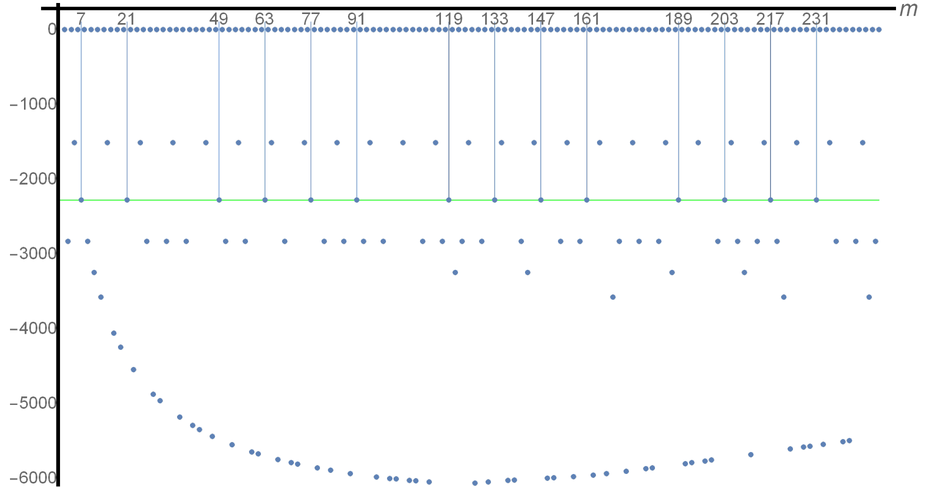
Finer Structure: Plot of $\log_{10} |\delta_{10001,3m} - \delta_{10001,3}|$



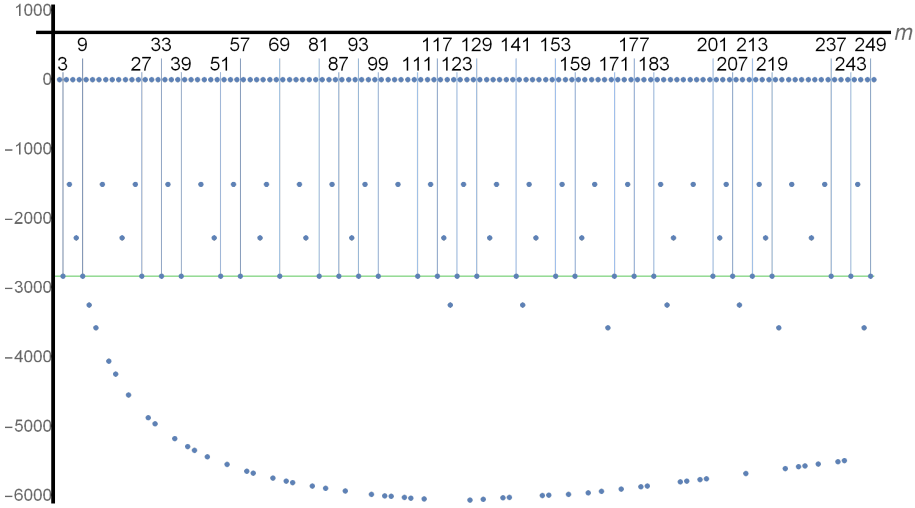
Finer Structure: Plot of $\log_{10}|\delta_{10001,3m} - \delta_{10001,3}|$



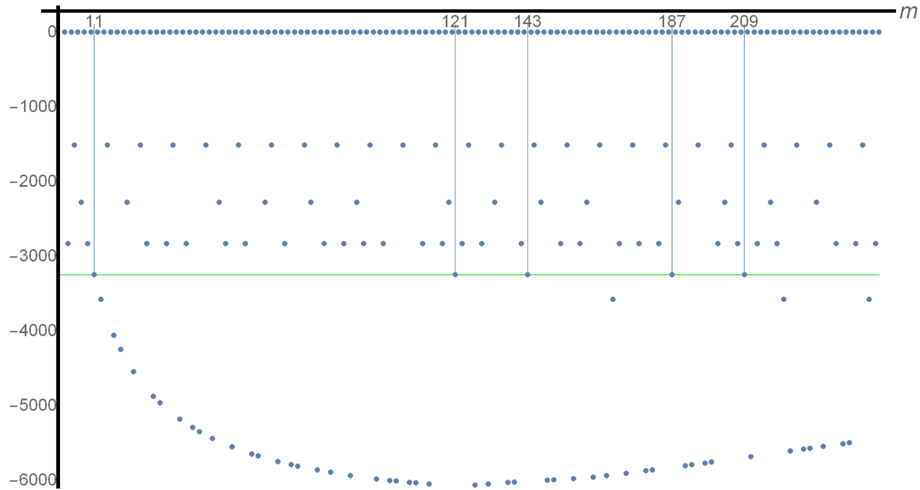
Finer Structure: Plot of $\log_{10}|\delta_{10001,3m} - \delta_{10001,3}|$



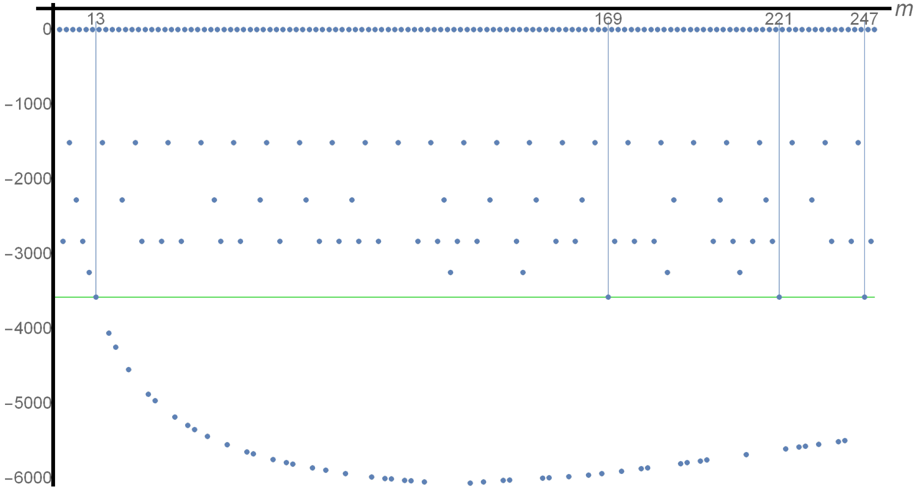
Finer Structure: Plot of $\log_{10}|\delta_{10001,3m} - \delta_{10001,3}|$



Finer Structure: Plot of $\log_{10}|\delta_{10001,3m} - \delta_{10001,3}|$

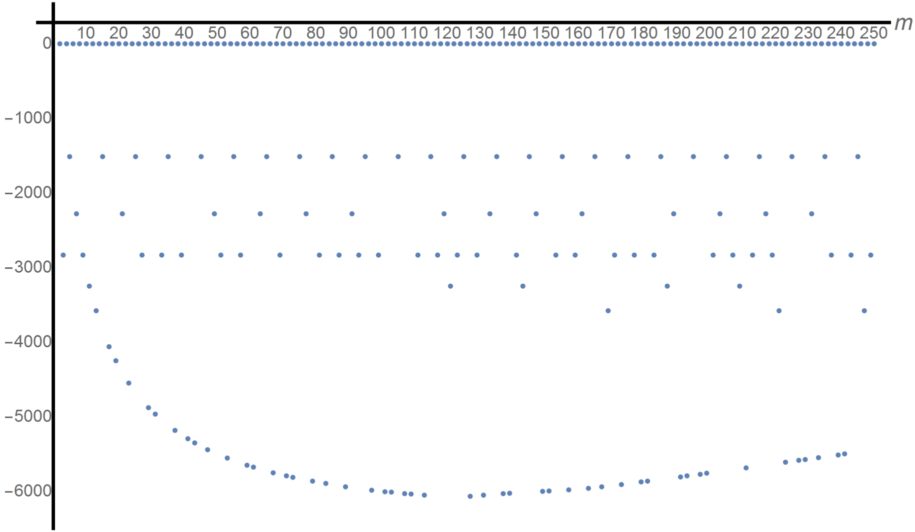


Finer Structure: Plot of $\log_{10} |\delta_{10001,3m} - \delta_{10001,3}|$



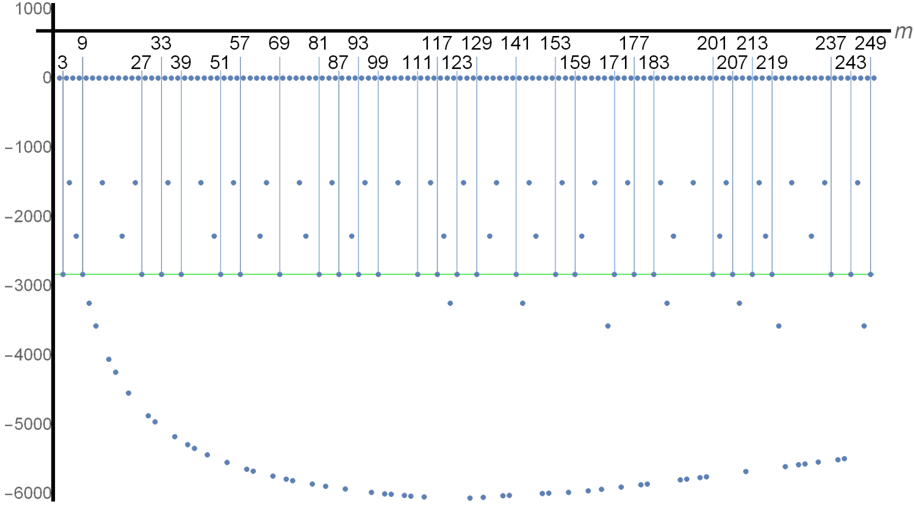
Finer Structure: Plot of $\log_{10} |\delta_{10001,3m} - \delta_{10001,3}|$

= Eratosthenes Sieve with primes order 2, 5, 7, 3, 11, 13, ...



Finer Structure: Plot of $\log_{10}|\delta_{10001,3m} - \delta_{10001,3}|$

= Eratosthenes Sieve with primes order 2, 5, 7, 3, 11, 13, ...



Expected Fractal Structure

Let n range over the arithmetical progression $d, 2d, \ldots, md, \ldots$ with

$$d = 2^{k_2} 3^{k_3} 5^{k_5} \ldots$$

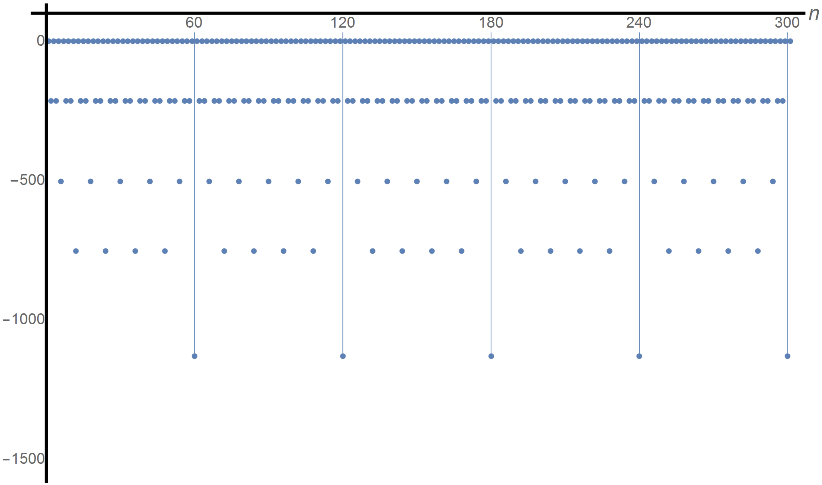
Corresponding Eratosthenes sublevel splits according to the divisibility of m by p_1, p_2, ... where these prime numbers are ordered in such a way that

$$p_1^{k_{p_1}+1} < p_2^{k_{p_2}+1} < \cdots < p_j^{k_{p_j}+1} < \ldots$$

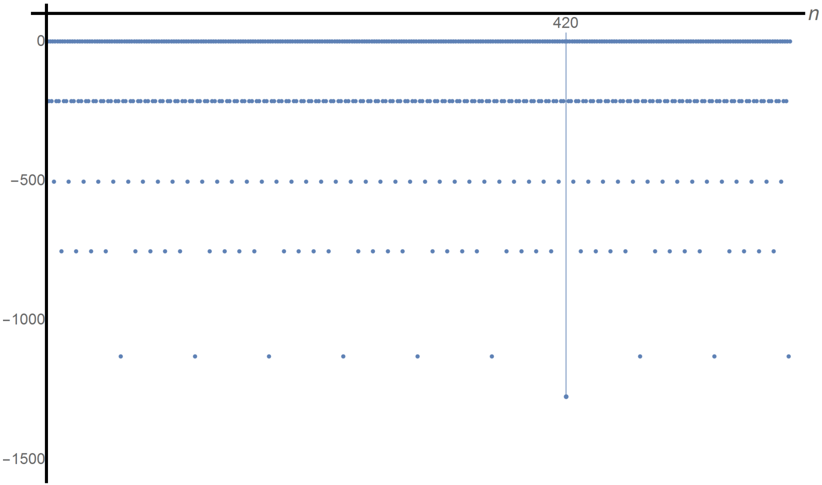
In the previous example $m = 3$, hence $k_2 = 0$, $k_3 = 1$, $k_5 = k_7 = \cdots = 0$ and $p_1 = 2$, $p_2 = 5$, $p_3 = 7$, $p_4 = 3$, $p_5 = 11$, $p_6 = 13$, ... according to

$$2^1 < 5^1 < 7^1 < 3^2 < 11^1 < 13^1 < \ldots$$

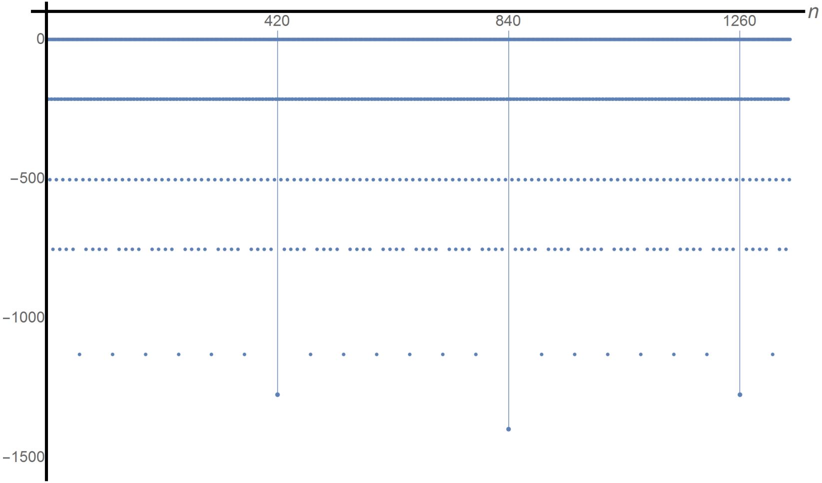
Dual sieve: Plot of $\log_{10}|\sum_{n=1}^{m} \delta_{N,n}|$ при $N = 5001$



Dual sieve: Plot of $\log_{10}|\sum_{n=1}^{m}\delta_{N,n}|$ при $N = 5001$



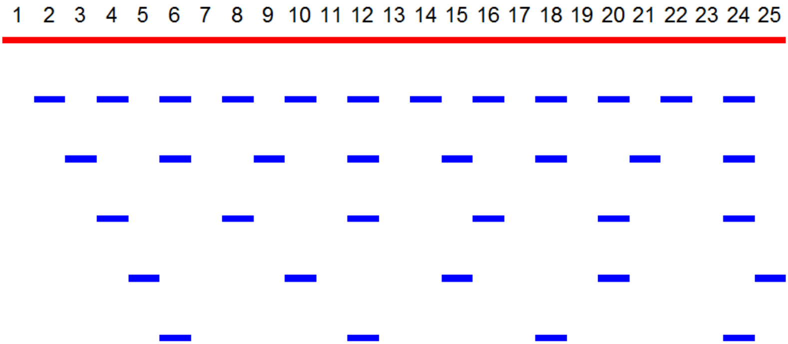
Dual sieve: Plot of $\log_{10}\left|\sum_{n=1}^{m}\delta_{N,n}\right|$ при $N = 5001$



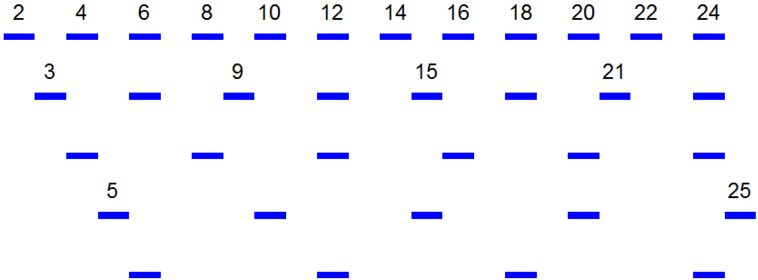
Inheritable divisor: $k_{\leq}|m \iff 1|m \;\&\; 2|m \;\&\; 3|m \;\&\; \ldots \;\&\; k|m$

Maximal inheritable divisor: $k_{\leq}||m \iff k_{\leq}|m \;\&\; (k+1)\nmid m$

Sieve of Eratosthenes

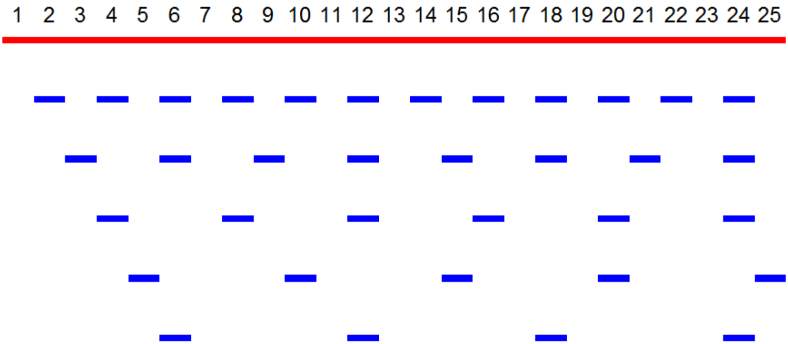


Sieve of Eratosthenes

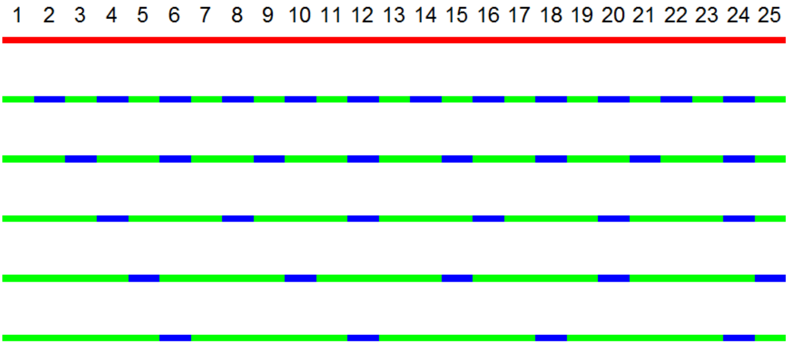


$$\ln|\delta_{N,n} - 1|$$

Sieve of Eratosthenes (repeated)



Sieve of Eratosthenes vs dual sieve



Dual sieve

Dual sieve

$$\ln \left| \sum_{n=1}^{m} \delta_{N,n} \right|$$

Davenport–Heilbronn function

$$f(s) = \sum_{n=1}^{\infty} d(n) n^{-s}$$

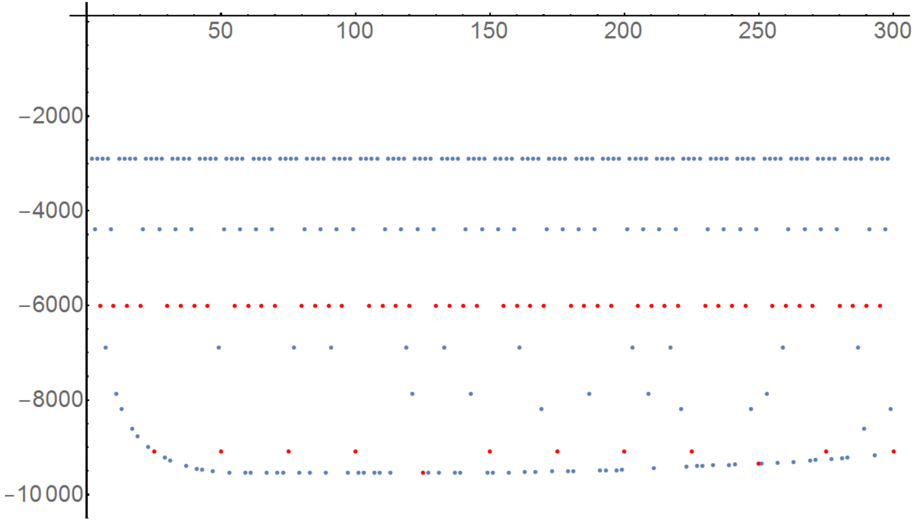
where

$$d(n) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 5 \\ 1, & \text{if } n \equiv 1 \pmod 5 \\ \tau, & \text{if } n \equiv 2 \pmod 5 \\ -\tau, & \text{if } n \equiv 3 \pmod 5 \\ -1, & \text{if } n \equiv 4 \pmod 5 \end{cases}$$

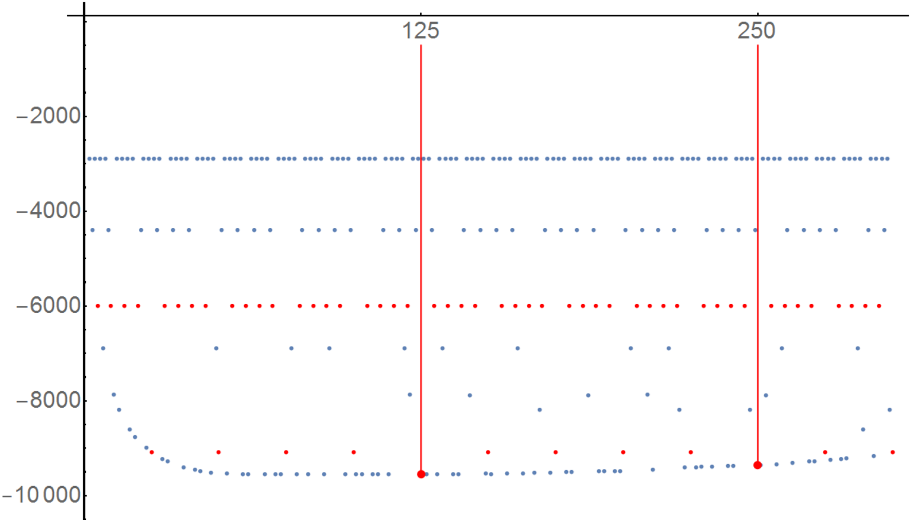
and

$$\tau = \frac{-2 + \sqrt{10 - 2\sqrt{5}}}{-1 + \sqrt{5}}$$

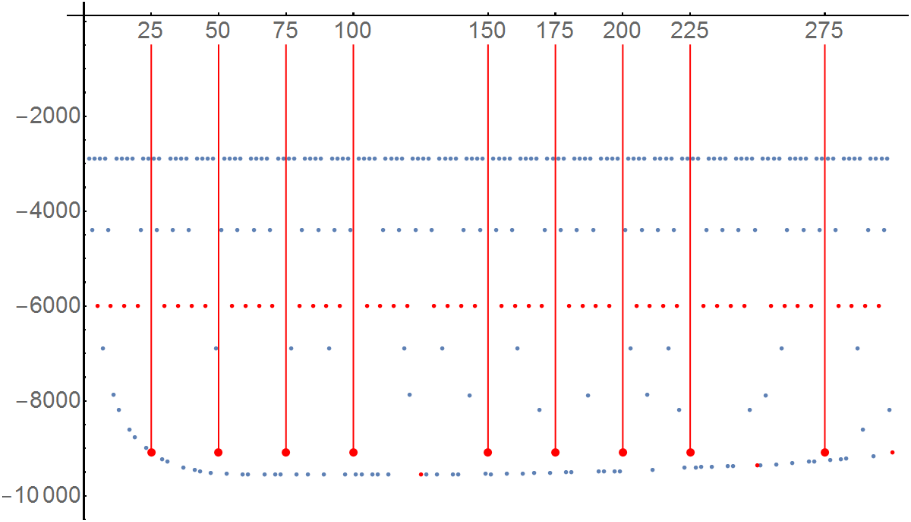
Sieve of Eratosthenes for $f(s)$



Sieve of Eratosthenes for $f(s)$



Sieve of Eratosthenes for $f(s)$



THANK YOU FOR ATTENTION!

`http://logic.pdmi.ras.ru/~yumat`