Sharp inequalities on BMO-space and Bellman function

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April 8, 2021

Colloquium

Departement of Mathematics and Computer Science

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$$\mathrm{BMO}_\varepsilon \stackrel{\mathrm{def}}{=} \{\varphi\in\mathrm{BMO}\colon \|\varphi\|\le\varepsilon\}.$$

Extremal problems and their Bellman functions

Extremal problem.

For a given real-valued function f on $\mathbb{R}$, maximize (or minimize) the value of the following integral functional

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over the ball BMO_ε.

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3. Boundary values:

$$\mathbf{B}(x_1, x_1^2) = f(x_1)\,.$$

Various choices of f

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$$f(s) = \chi_{(-\infty,-\lambda)\cup(\lambda,\infty)}(s),$$
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- L^p-estimates, in particular, equivalence of different BMO-norms:

$$f(s) = |s|^p, \qquad \mathbf{B}(x;p,\varepsilon) = \sup_{\varphi}\{\langle|\varphi|^p\rangle_J\}\,.$$

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The Bellman function is a solution of the boundary value problems for this equation:

$$\mathbf{B}(x_1, x_1^2) = f(x_1)\,.$$

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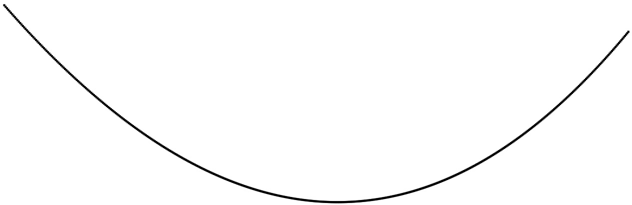
Properties of the Monge–Ampère foliation

- If two extremal lines intersect at a point, then **B** is linear
- If an extremal line intersects the upper boundary $\{x\colon x_2 = x_1^2 + \varepsilon^2\}$, then it touches it tangentially.

The left tangent foliation



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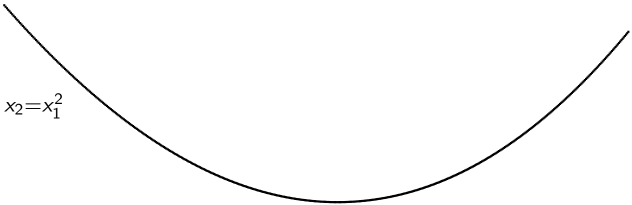

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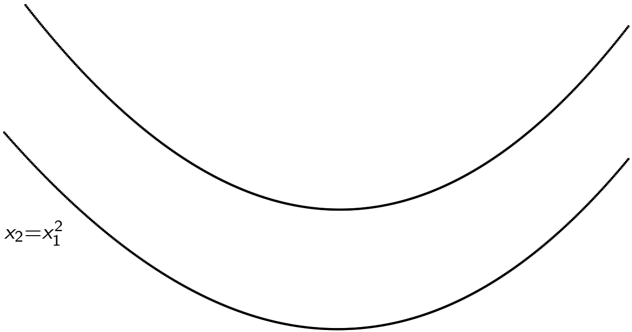

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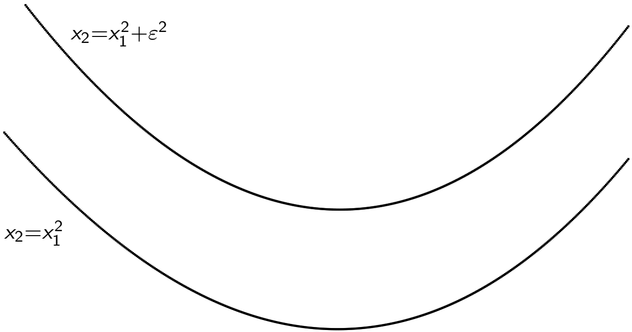

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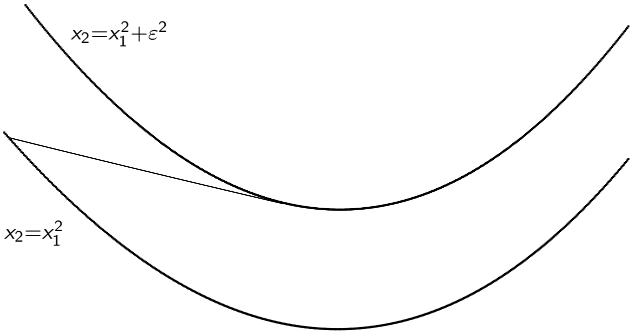

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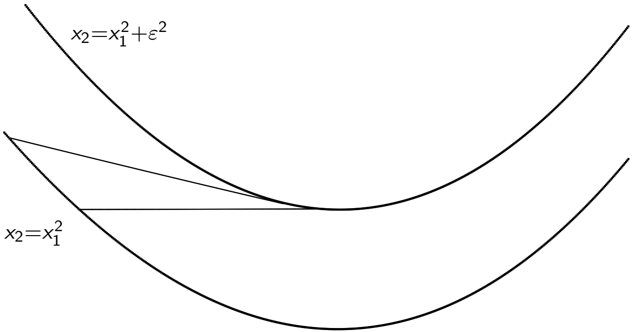

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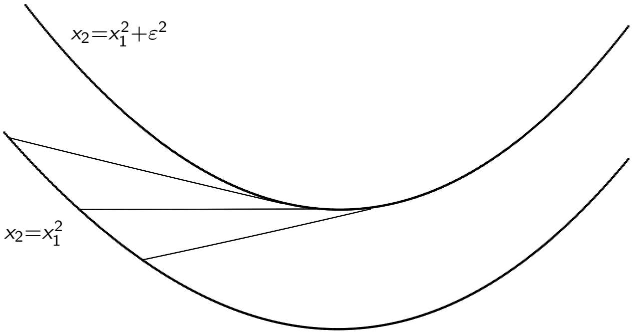

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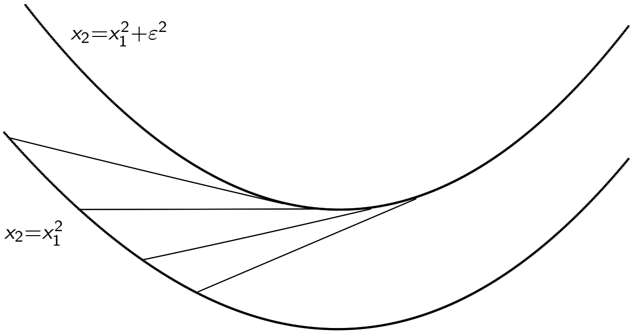

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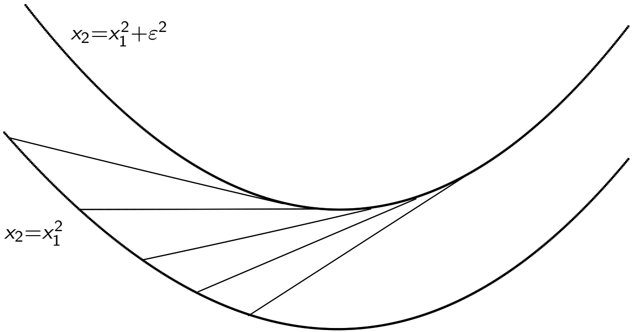

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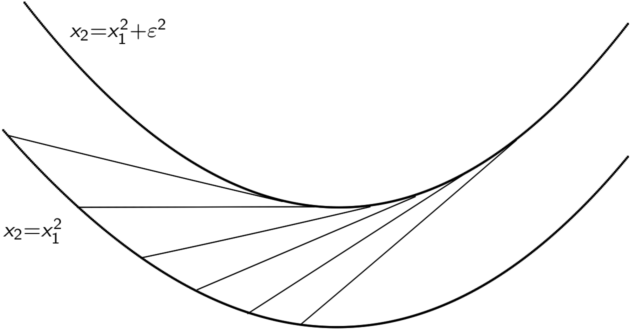

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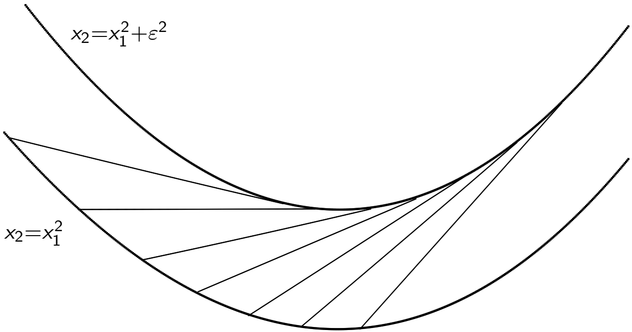

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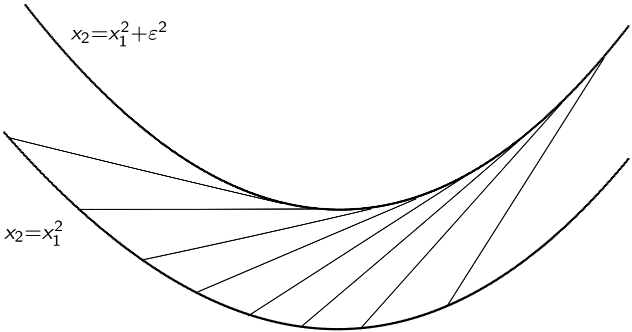

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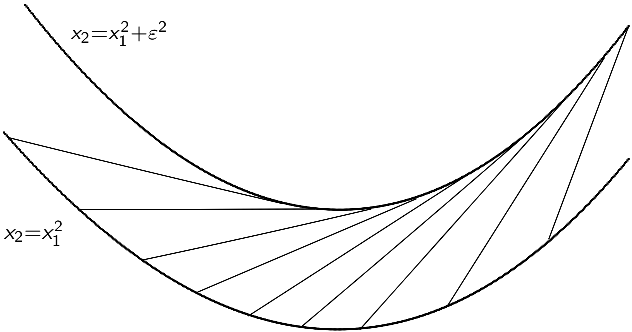

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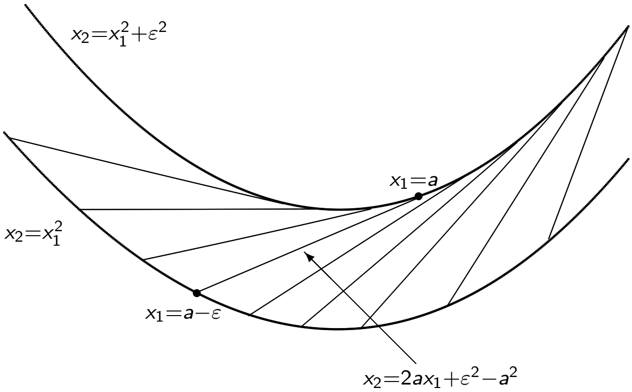
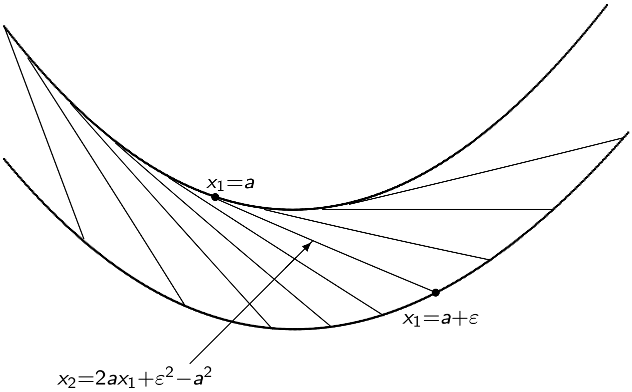


The right tangent foliation



An examples for the left foliation

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The corresponding Bellman function is

$$\mathbf{B}(x) = \frac{1 - \sqrt{\varepsilon^2 - x_2 + x_1^2}}{1 - \varepsilon} \exp\left\{x_1 + \sqrt{\varepsilon^2 - x_2 + x_1^2} - \varepsilon\right\} .$$

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It gives the sharp constants in the integral form of the John–Nirenberg inequality:

Theorem

If $\|f\| < 1$, *then*

$$\left\langle e^f \right\rangle_J \leq \frac{e^{-\|f\|}}{1 - \|f\|}\, e^{\langle f \rangle_J}.$$

The constants are sharp.

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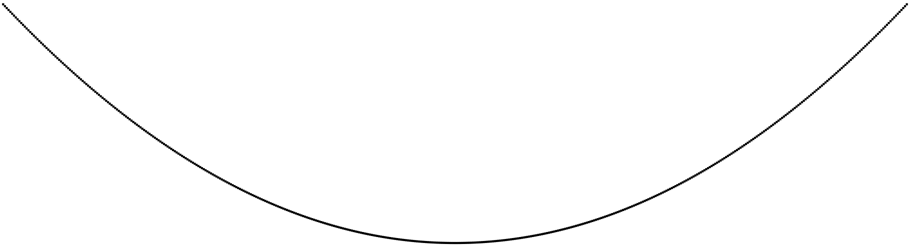
Any smooth function f with $f''' > 0$ produces the left foliation.

Any smooth function f with $f''' < 0$ produces the right foliation.

A cup



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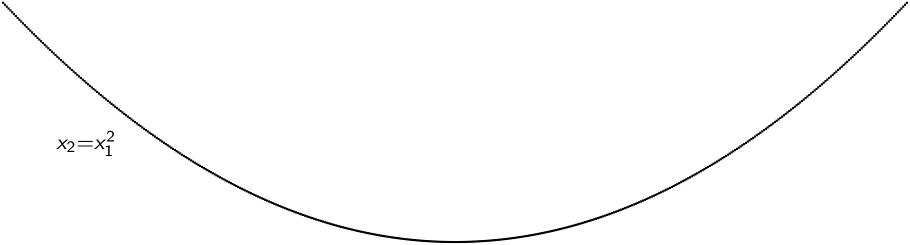

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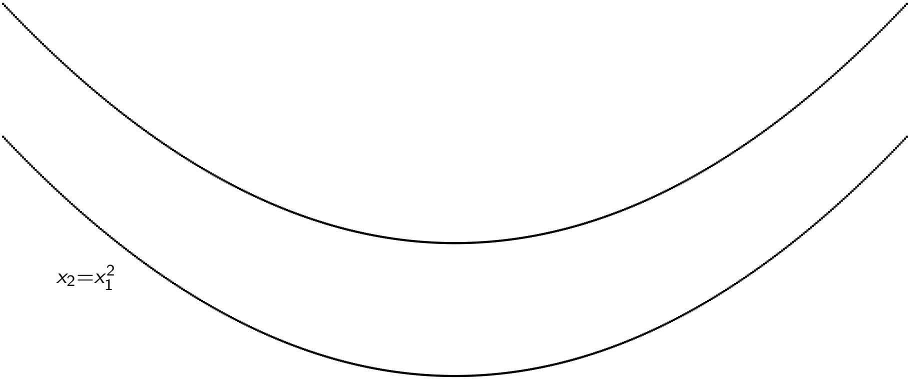

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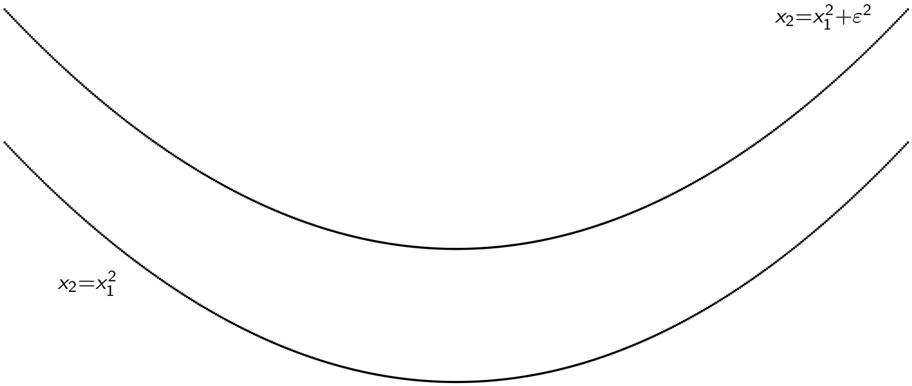

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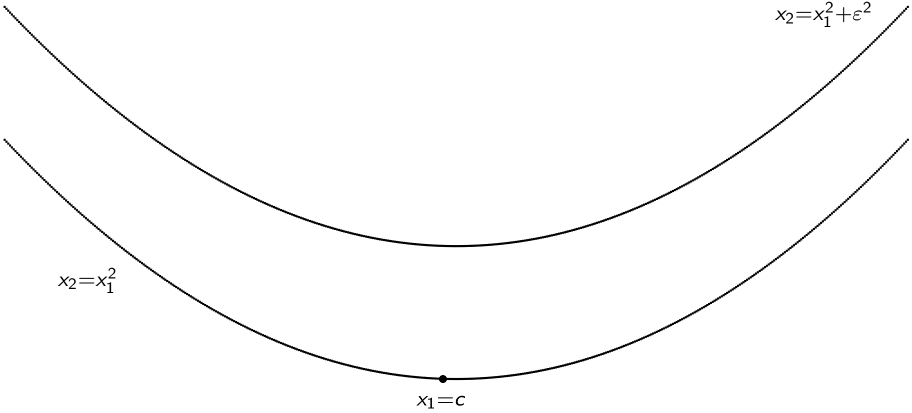

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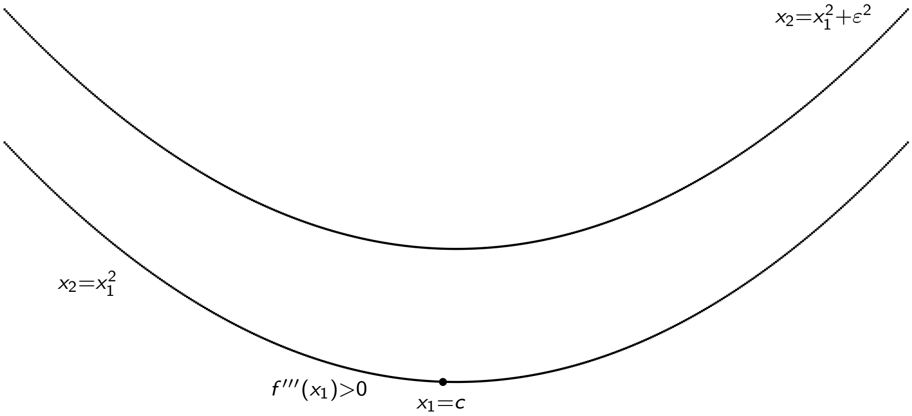

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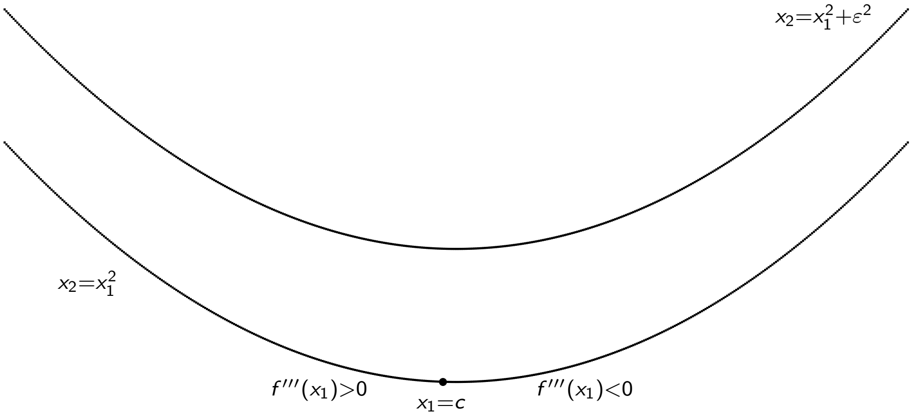

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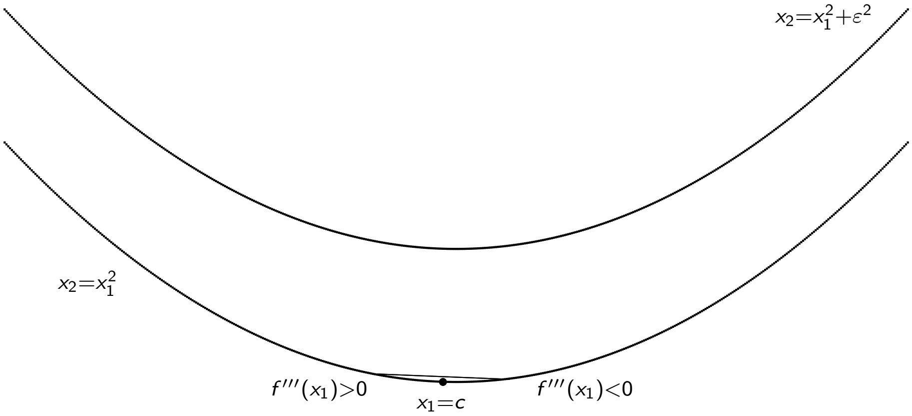

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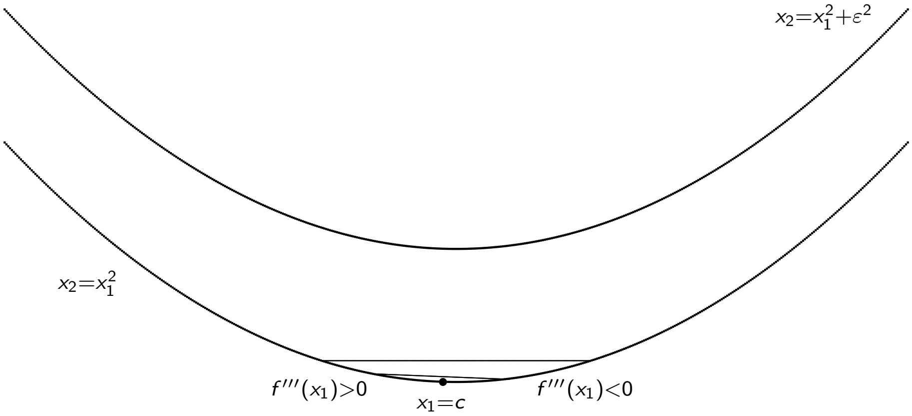

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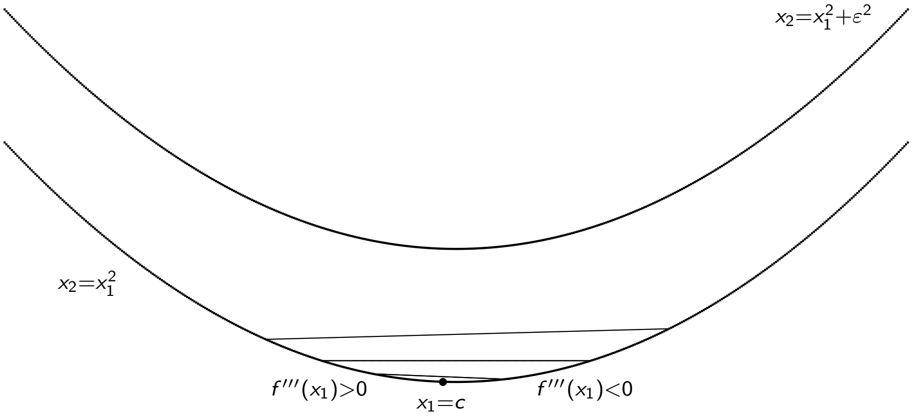

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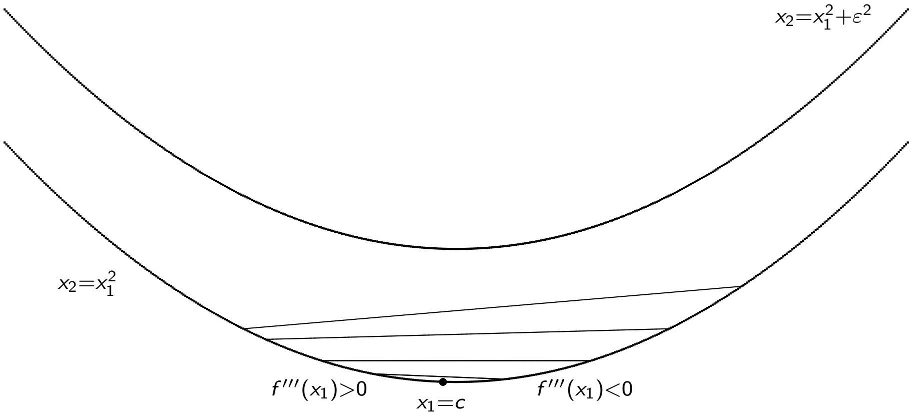

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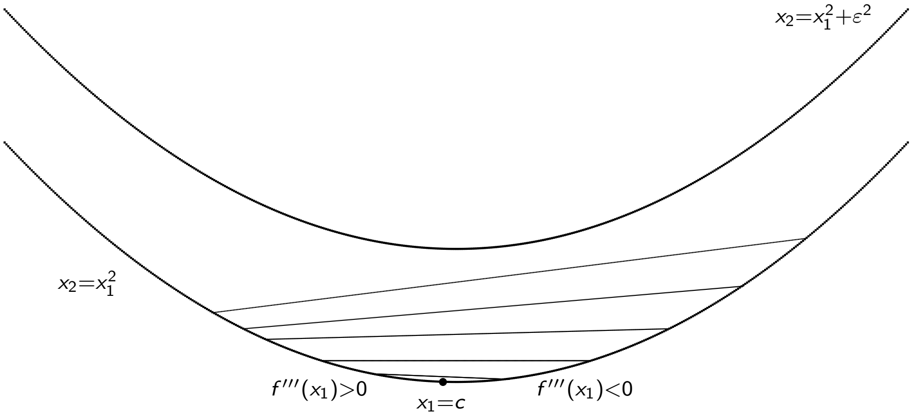

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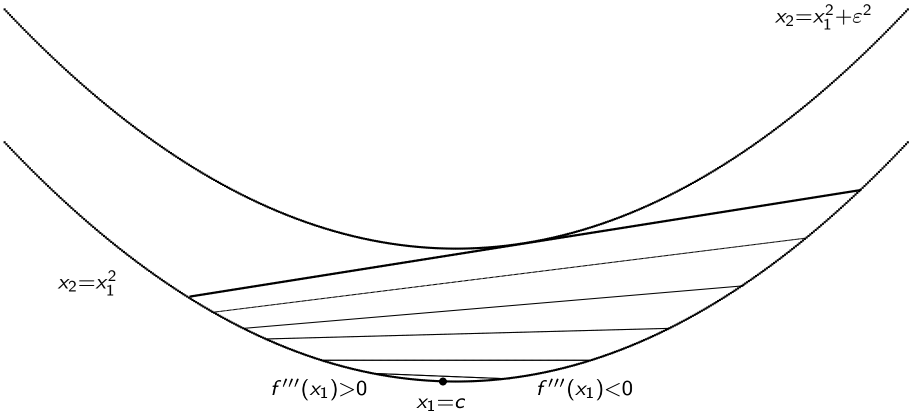

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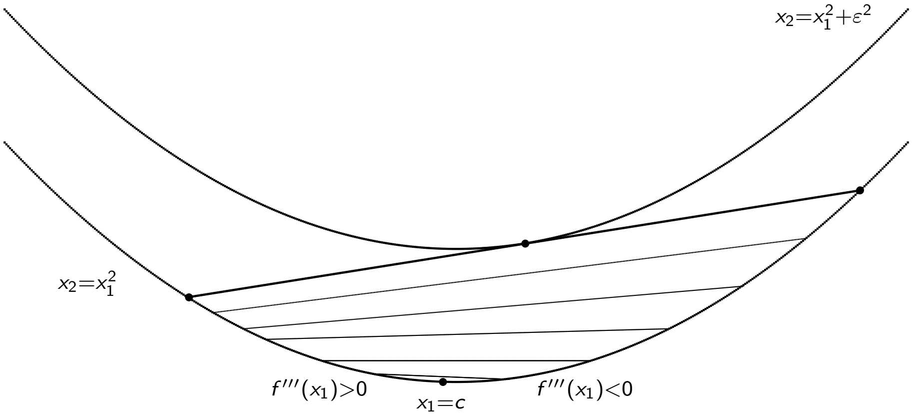

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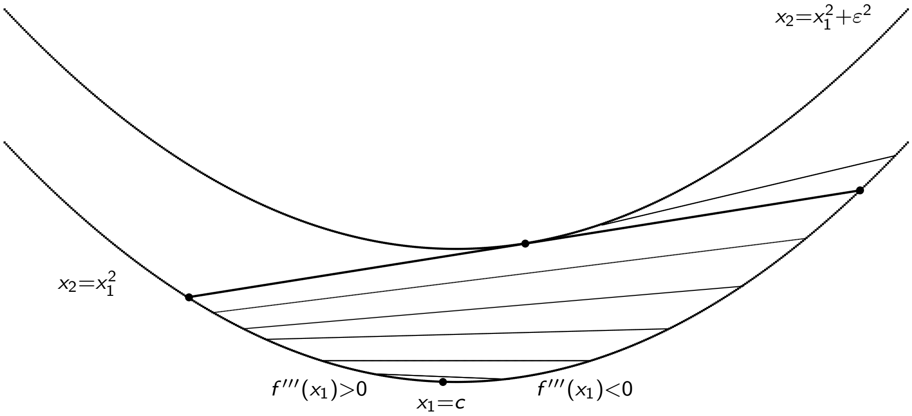

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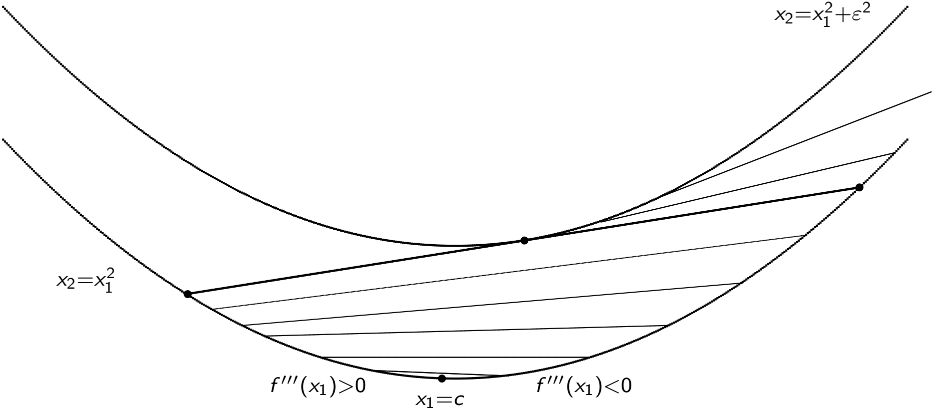

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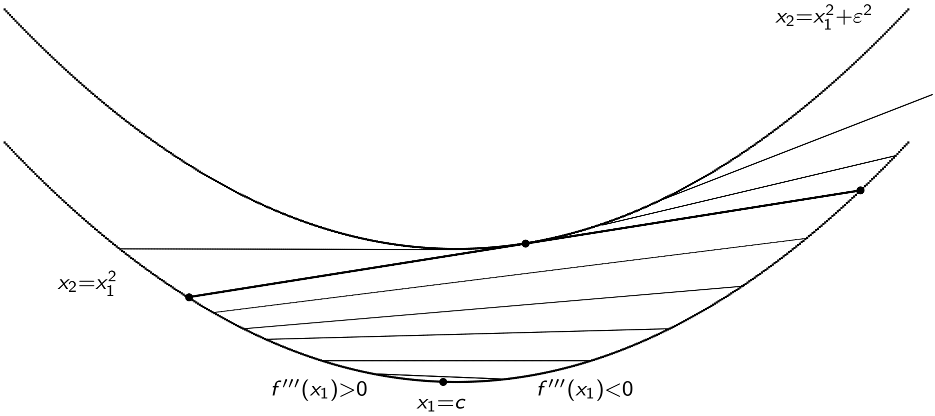

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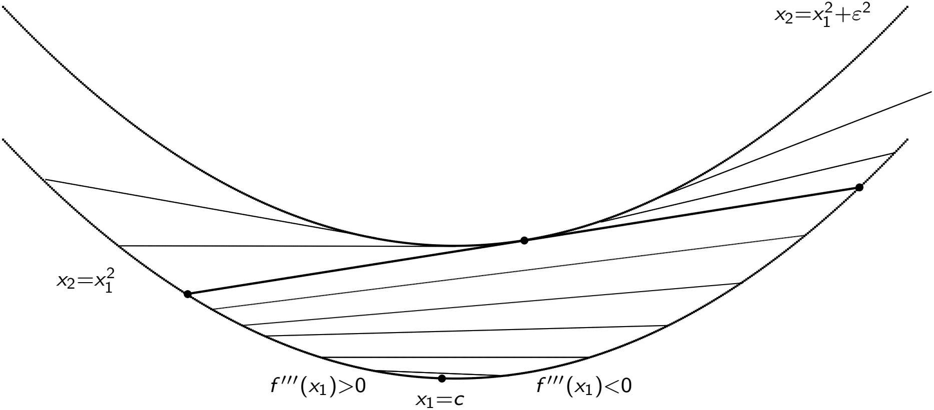

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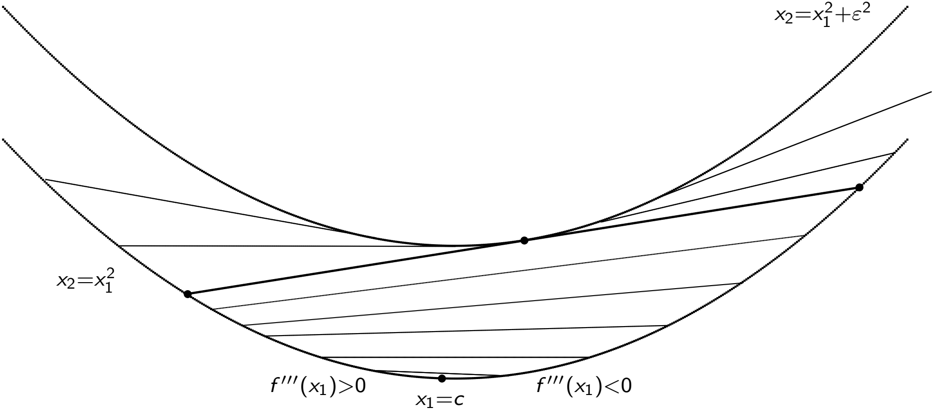


An example

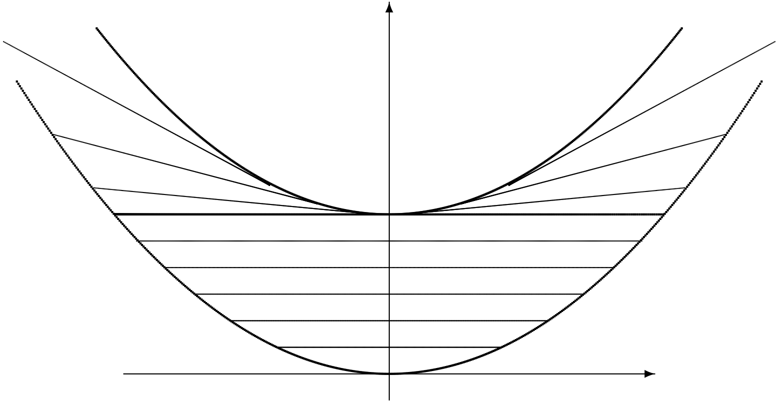
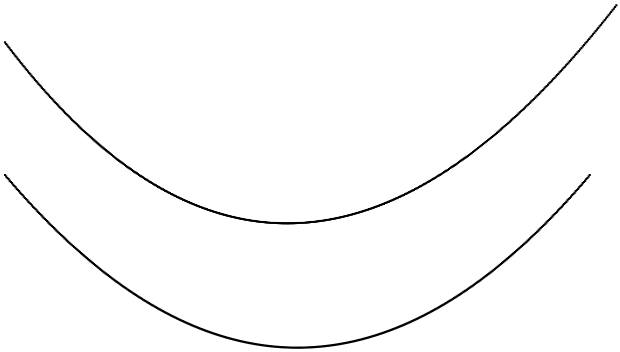


Figure: A cup gluing left and right tangent foliations.
Example: $f(s) = |s|^p$, $1 \le p < 2$.

Gluing left and right foliations in the reverse order



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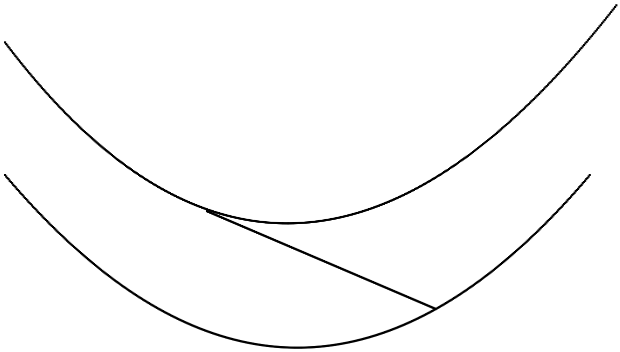

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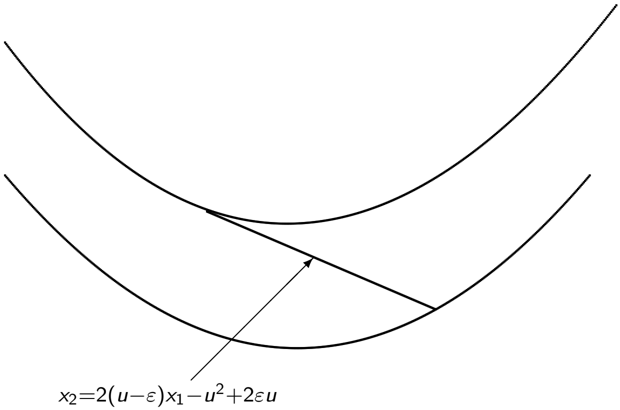

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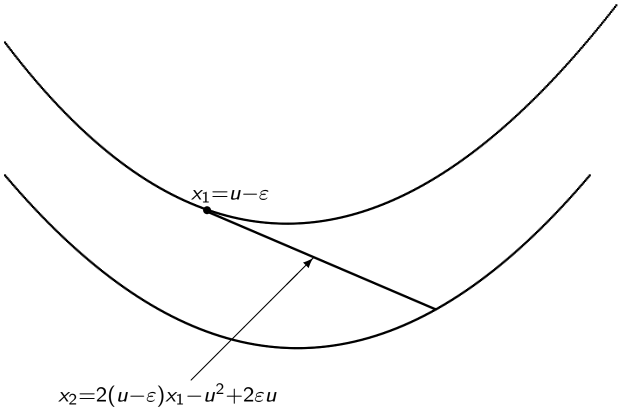

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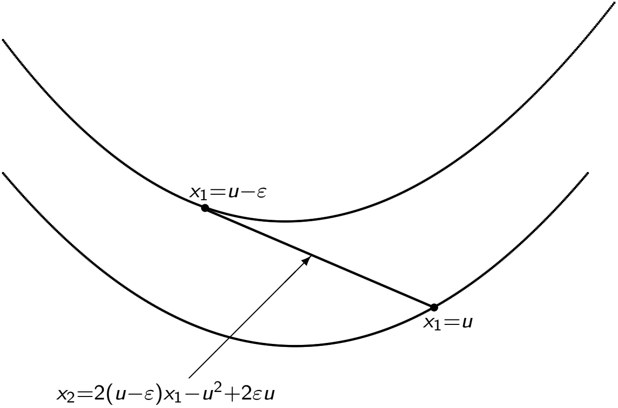

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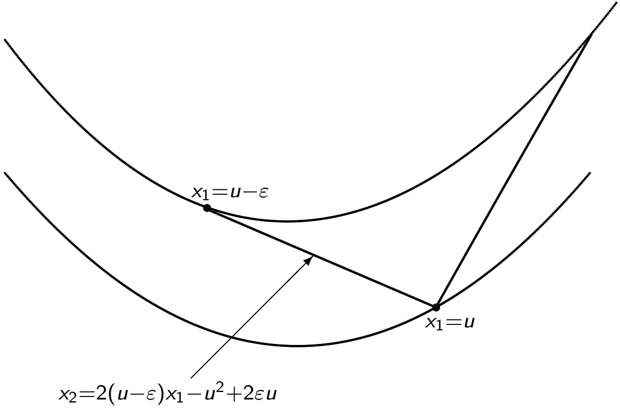

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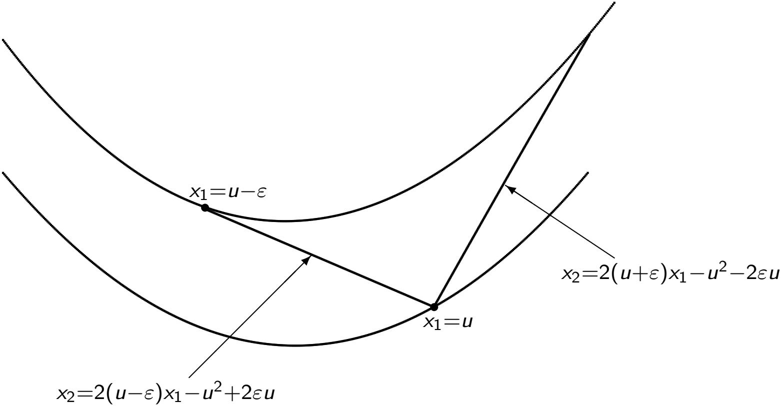

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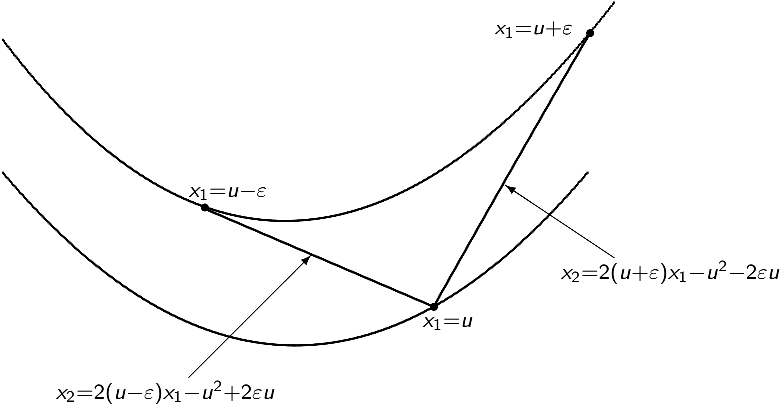

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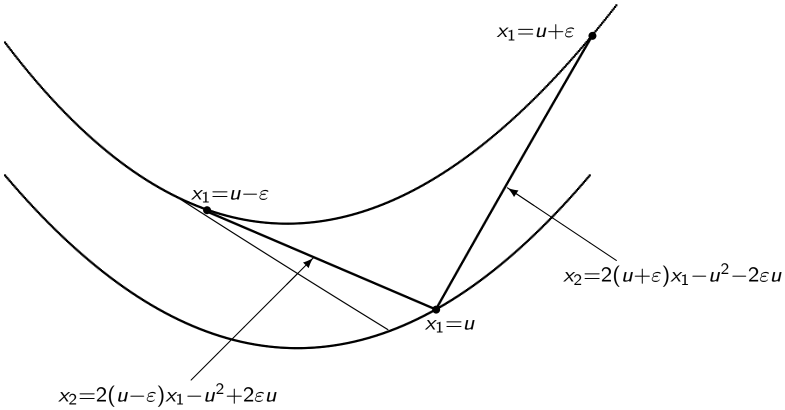

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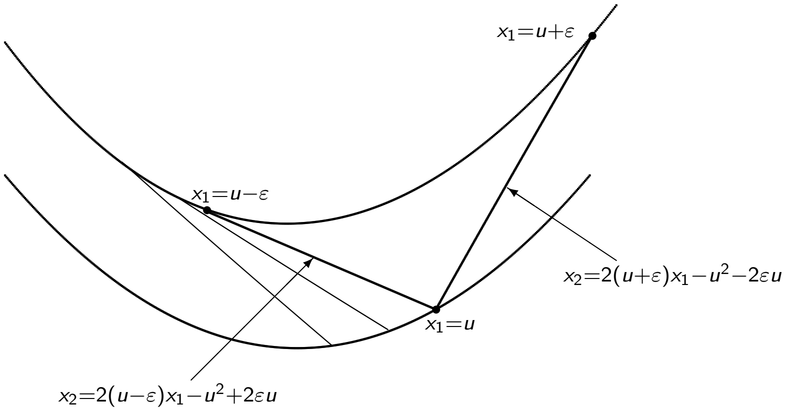

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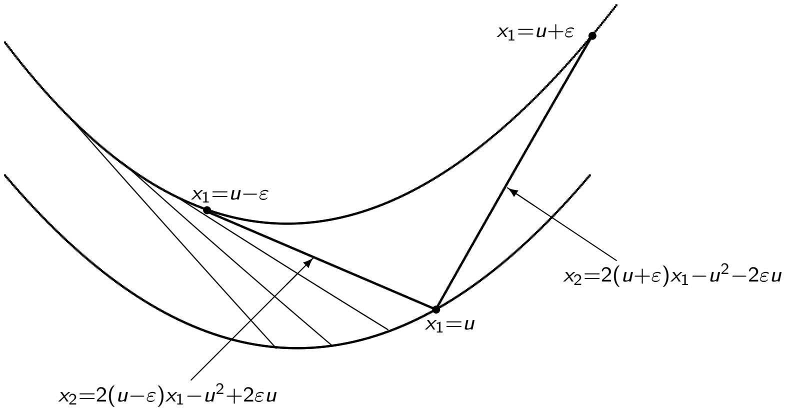

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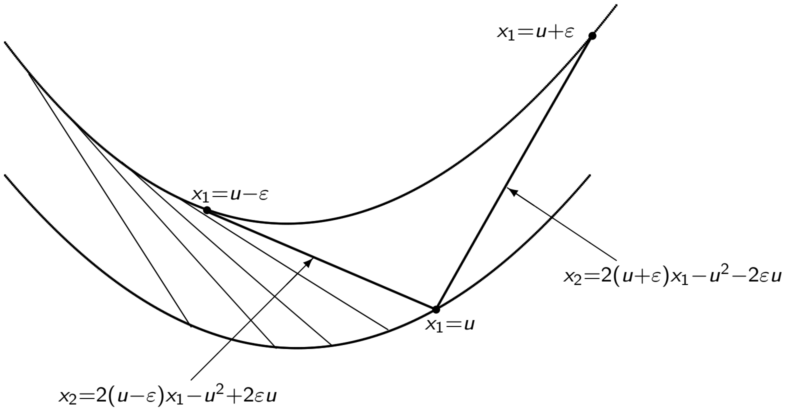

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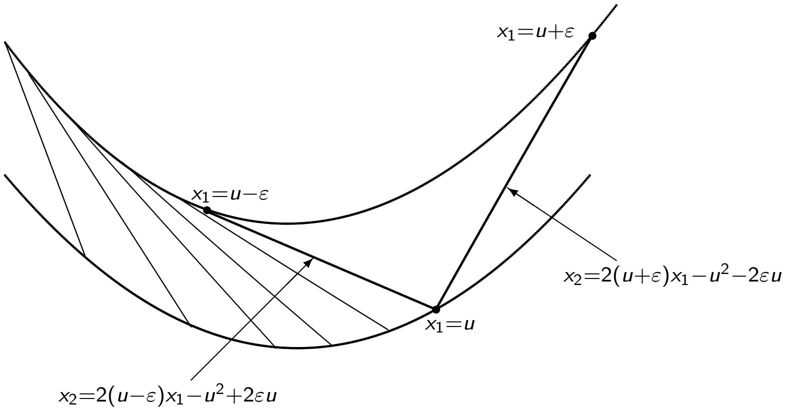

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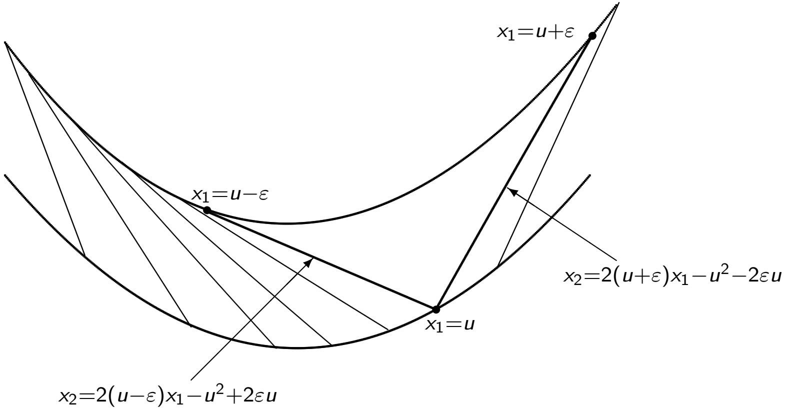

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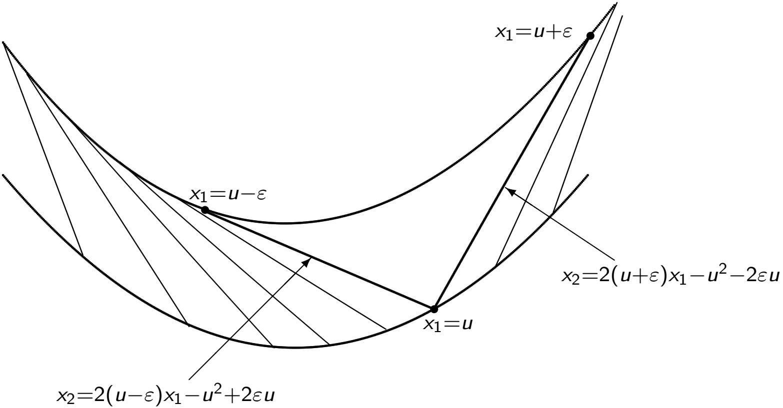
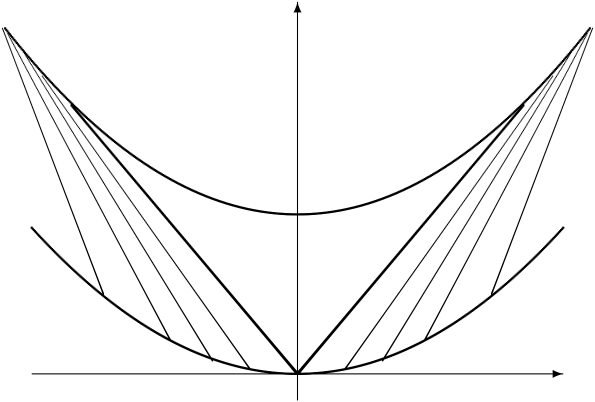


Figure: A triangle gluing right and left tangent foliations.

An example of gluing by a triangle



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Example: $f(s) = |s|^p$, $p > 2$.

A more difficult example of foliation

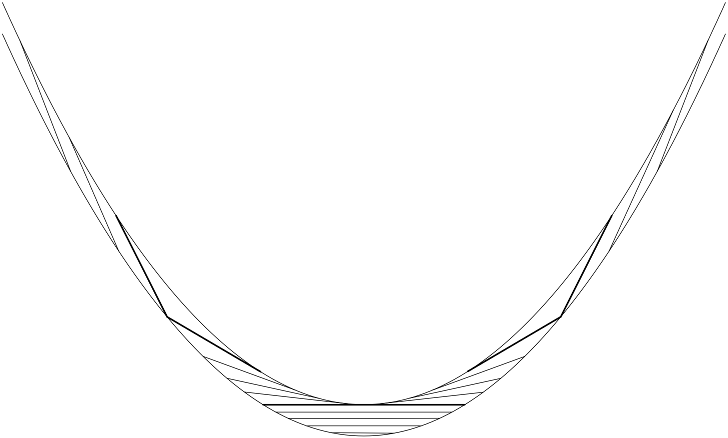
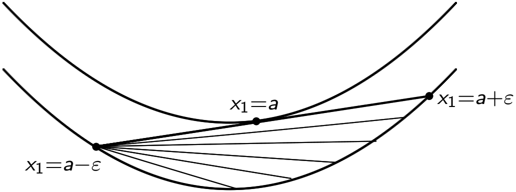
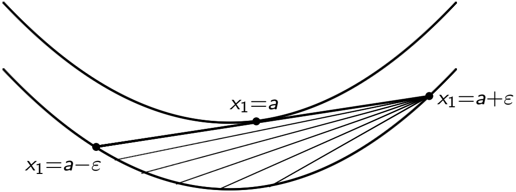


Figure: The Monge–Ampère foliation for $f(s) = |s|^p$, $0 < p < 1$.

A cup: singular foliations





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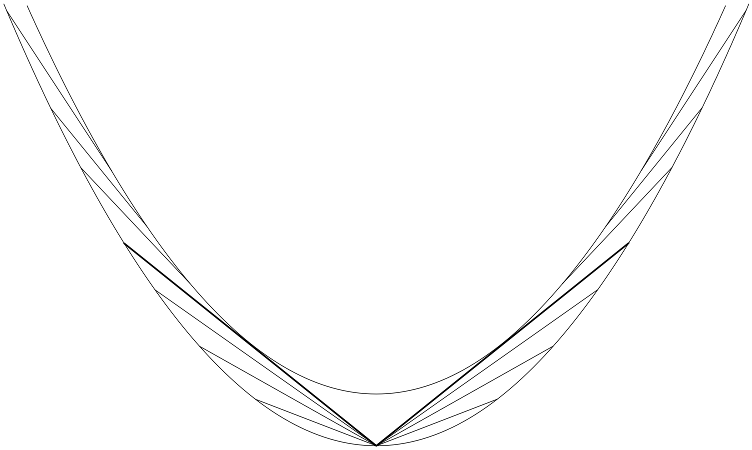

Figure: The Monge–Ampère foliation for $f(s) = -|s|^p$, $0 < p < 1$.

Equivalence of different BMO-norms

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A bit more difficult situation

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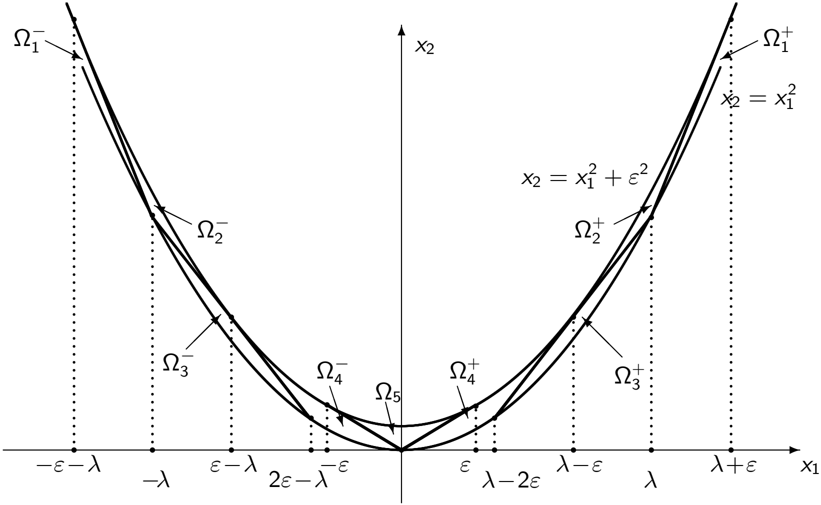

Figure: $f(s) = 1$ for $|s| > \lambda$ and $f(s) = 0$ for $|s| < \lambda$

Explicit Bellman function: the classical form

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$$\mathbf{B}(x_1, x_2; \lambda, \varepsilon) = \frac{e}{2}\left(1 - \sqrt{1 - \frac{x_2 - x_1^2}{\varepsilon^2}}\right) \exp\left\{\frac{|x_1| - \lambda}{\varepsilon} + \sqrt{1 - \frac{x_2 - x_1^2}{\varepsilon^2}}\right\};$$

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- *if* $x \in \Omega_5$*, then* $\mathbf{B}(x_1, x_2; \lambda, \varepsilon) = \dfrac{x_2}{4\varepsilon^2}\exp\left\{2 - \dfrac{\lambda}{\varepsilon}\right\}.$

Classical form of the John–Nirenberg inequality

Theorem

$$\Big|\Big\{t\in J\colon\ |f(t)-\langle f\rangle_{_J}|\ge\lambda\Big\}\Big|\le\frac{e^2}{4}\exp\Big\{-\frac{\lambda}{\|f\|}\Big\}|J|.$$

for all functions $f\in\mathrm{BMO}(J)$. *All constants are sharp.*

Extremal problem for two functionals

For two given real-valued function f and g on $\mathbb{R}$, maximize (or minimize) the value of the following integral functional

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over the ball $\mathrm{BMO}_\varepsilon(J)$ assuming the value the second functional $\langle g(\varphi)\rangle_J$ to be fixed.

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How to find the Bellman function

Now the Bellman function is defined on the following three dimensional domain:

$$\Omega = \{x = (x_1, x_2, x_2) \colon\ x_1^2 \le x_2 \le x_1^2 + \varepsilon^2,\ \mathbf{B}_g^{\min}(x_1, x_2) \le x_3 \le \mathbf{B}_g^{\max}(x_1, x_2)\}\,.$$

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So, to find the function $\mathbf{B}$ we look for the minimal locally concave function on Ω with the given boundary values.

A triangle gluing right and left tangent foliations

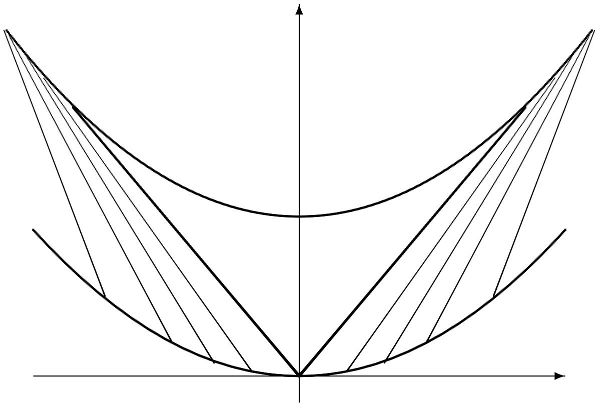

Figure: Foliation for $\mathbf{B}_g^{\max}$, if $g(t) = |t|^p$, $p > 2$.

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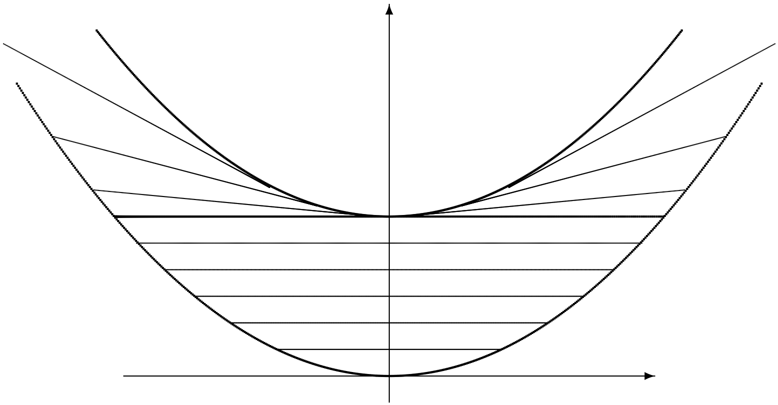

Figure: Foliation for $\mathbf{B}_g^{\min}$, if $g(t) = |t|^p$, $p > 2$.

Foliation for **B**

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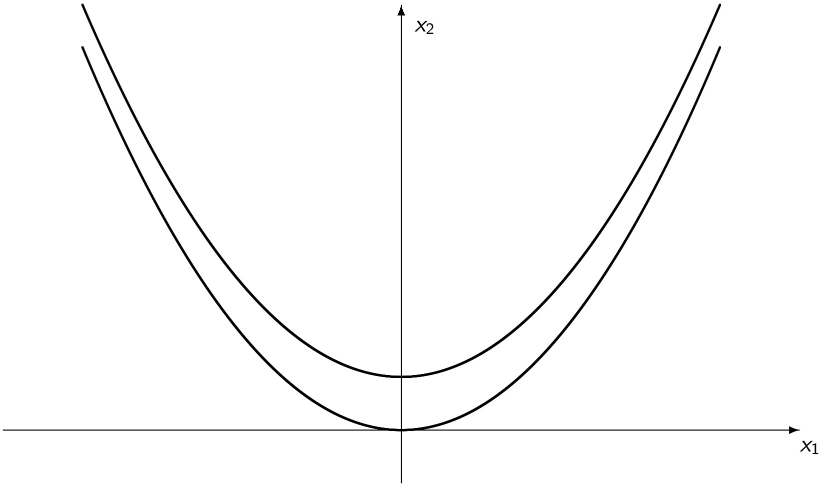

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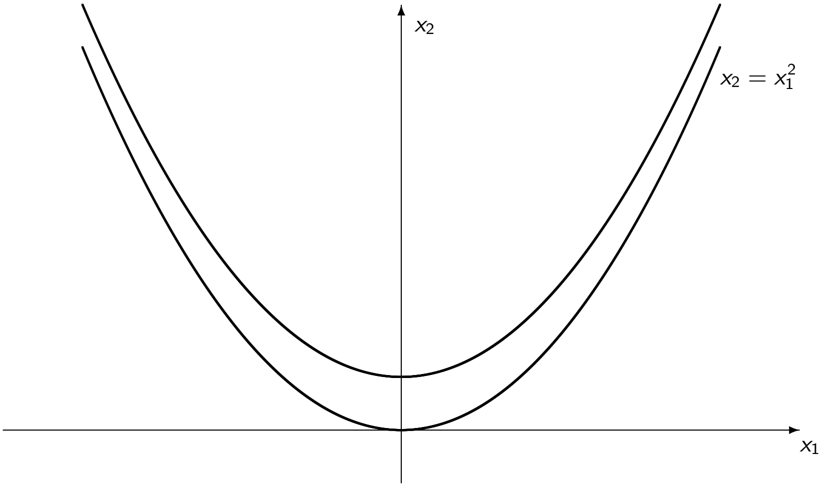

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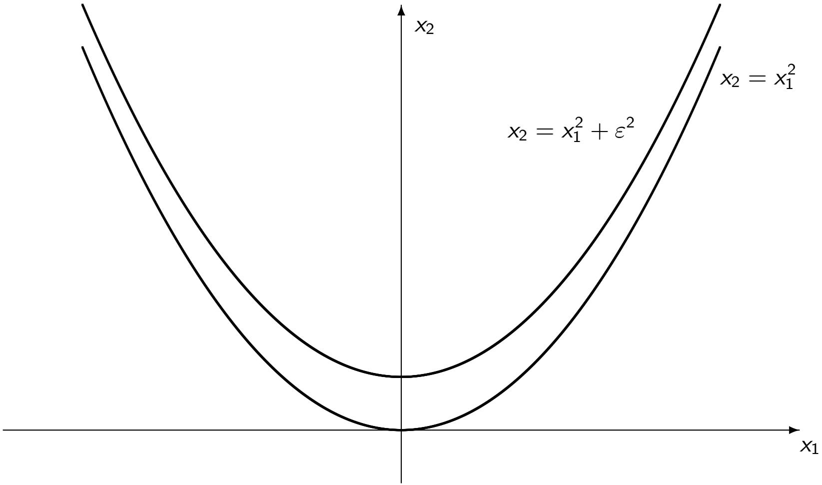

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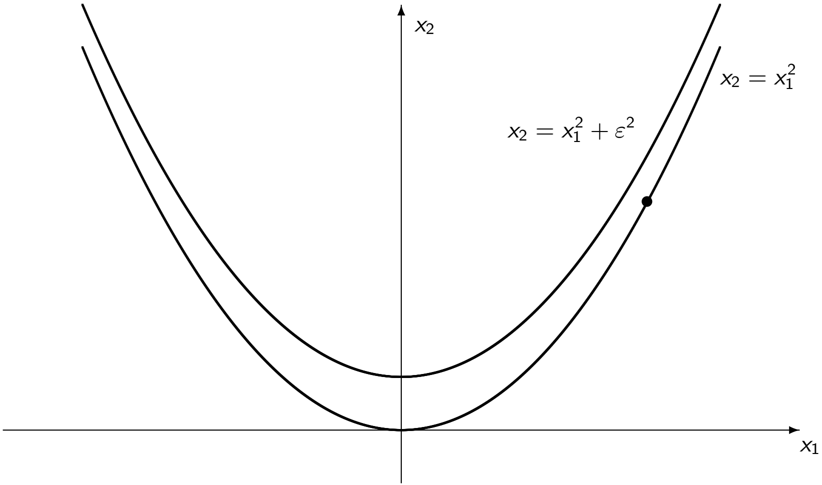

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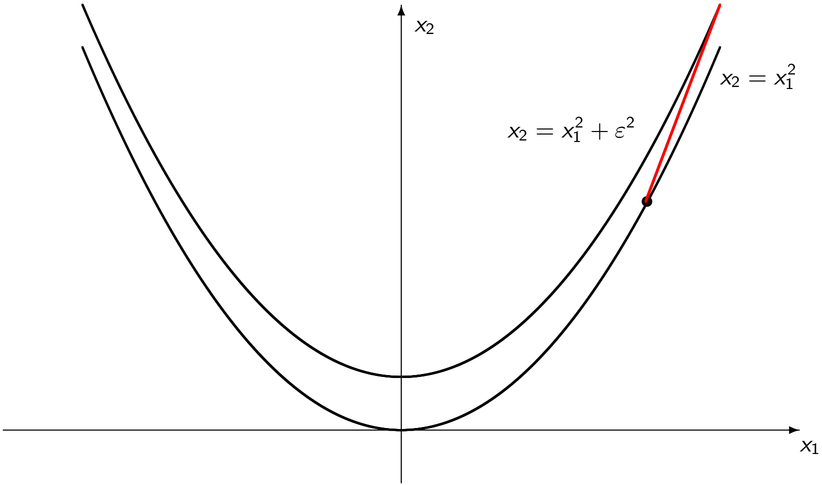

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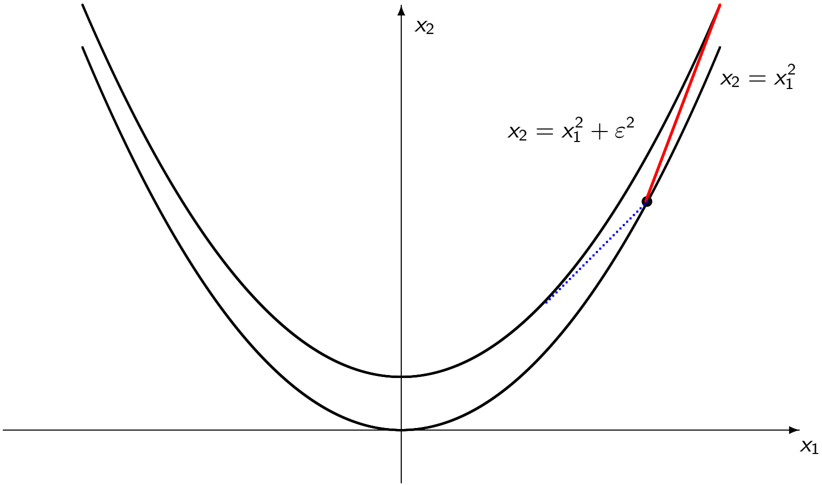

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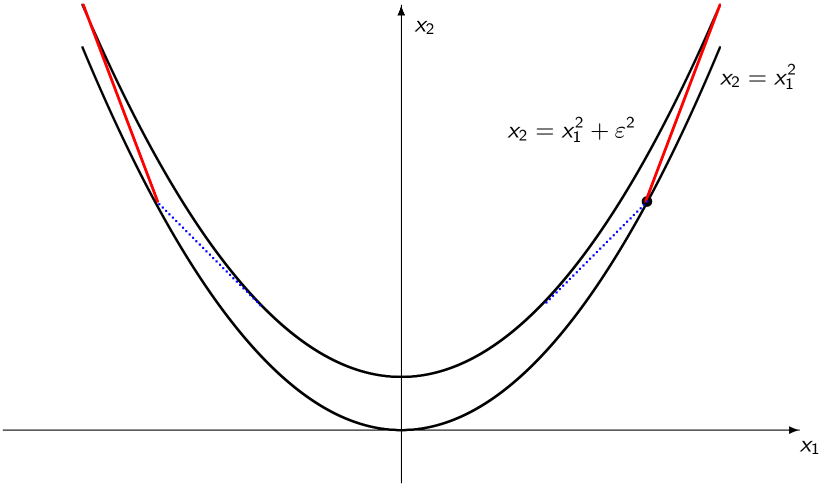
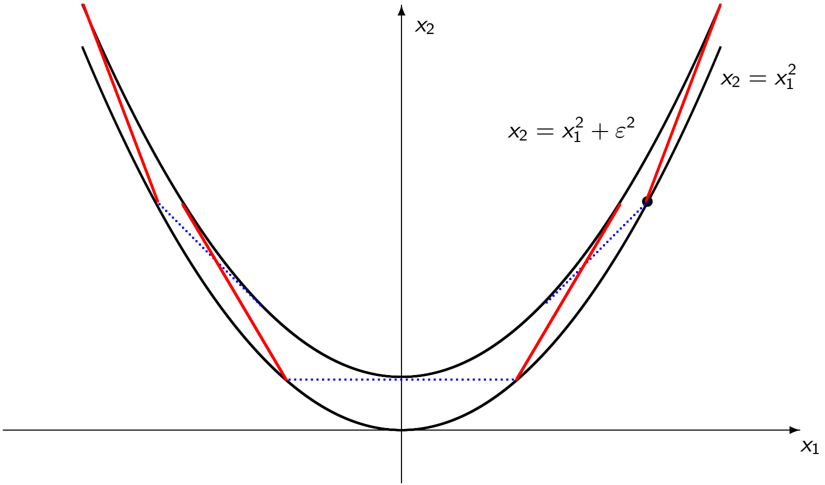
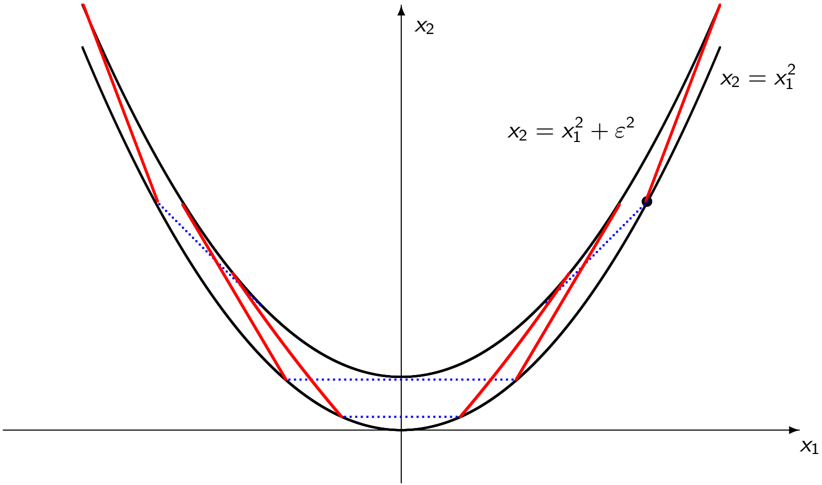


Foliation for **B**



Foliation for B



How to find the sharp constant

After the Bellman function $\mathbf{B}$ with $f(t) = |t|^r$ and $g(t) = |t|^p$ is found, we are able to calculate the sharp constant as follows:

$$C(p,r) = \sup_{(0,x_2,x_3)\in\Omega} \frac{\mathbf{B}(0,x_2,x_3;\varepsilon)}{x_3}\,.$$

Multiplicative inequality

Theorem

For any interval $I \in \mathbb{R}$ the inequality

$$\|\varphi\|^r_{L^r(I)} \le C(p,r)\cdot\|\varphi\|^p_{L^p(I)}\cdot\|\varphi\|^{r-p}_{\mathrm{BMO}(I)}, \qquad \int\limits_I \varphi(t)\,dt = 0, \quad 1 \le p \le r < \infty,$$

holds with the sharp constant

$$C(p,r) = \frac{\Gamma(r+1)}{\Gamma(p+1)} \qquad \text{if} \quad r > 2\,.$$