Department of Mathematics and Computer Science
@ St.Petersburg State University

COLLOQUIUM

Thursday, October 21, 17:15

Zoom 958-115-833, room 105 (14th line V.O., 29)

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Mean distance between two random points

One of the classical Sylvester questions asks for the probability that four points X_1, X_2, X_3, X_4 independently and uniformly distributed in some plane convex figure $K \subset \mathbb{R}^2$ form a triangle. Blaschke answered it showing that for any convex figure K

$$\frac{35}{12\pi^2} \leq \boldsymbol{P}[conv(X_1, X_2, X_3, X_4) \text{is triangle}] \leq \frac{1}{3}. \qquad (1)$$

The lower bound is achieved if and only if $K$is an ellipse, and the upper one if and only if K is a triangle. It is straightforward that $\boldsymbol{P}[conv(X_1, X_2, X_3, X_4)\text{is triangle}] = 4\frac{\boldsymbol{E}S(conv(X_1,X_2,X_3))}{S(K)}$, where $S(\cdot)$ denotes the area of a plane figure. Therefore (1) is equivalent to

$$\frac{35}{48\pi^2} \leq \frac{\boldsymbol{E}S\big(conv(X_1, X_2, X_3)\big)}{S(K)} \leq \frac{1}{12},$$

which gives the optimal lower and upper bounds for the normalized average area of the random triangle inside a convex figure. If we have only two random points inside K, they form a random segment with some random length. Thus, it is natural to ask for the optimal bounds of the normalized average length of this segment. We will show that for any convex figure $K \subset \mathbb{R}^2$ with non-empty interior,

$$\frac{7}{60} < \frac{\boldsymbol{E} \parallel X_1 - X_2 \parallel}{P(K)} < \frac{1}{12},$$

where $P(K)$ denotes the perimeter of K. The both lower and upper bounds are optimal. Based on the joint paper: G. Bonnet, A. Gusakova, Ch. Thäle, and D. Zaporozhets, “Sharp inequalities for the mean distance of random points in convex bodies”, Adv. Math., 386 (2021).

Everyone is welcome!