A note on a special measure

Dmitriy M. Stolyarov

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Let $\Lambda$ be the light cone

$$\Lambda = \{ x \in \mathbb{R}^4 \mid x_1^2 = x_2^2 + x_3^2 + x_4^2 \}.$$  

This cone has many famous properties. One of them was found by Kowalski and Preiss [1]: the Lebesgue measure $\lambda$ on this cone is uniform, that is, for any Euclidean ball $B_r(x)$ of radius $r$ centered at $x \in \Lambda$, the value $\lambda(B_r(x))$ is independent of $x$. They found exact expressions for the values $\lambda(B_r(x))$. We will provide a different computation, which leads to the same result. First, for any Schwartz function $\Phi$ on $\mathbb{R}^4$,

$$\int_{\mathbb{R}^4} \Phi(x) \, d\lambda(x) =$$

$$\sqrt{2} \iint_{\mathbb{R}^3} \left( \Phi \left( x_1, x_2, x_3, \sqrt{x_1^2 + x_2^2 + x_3^2} \right) + \Phi \left( x_1, x_2, x_3, -\sqrt{x_1^2 + x_2^2 + x_3^2} \right) \right) \, dx_1 \, dx_2 \, dx_3 \quad (1)$$

since the gradient of the function $(x_1, x_2, x_3) \mapsto \sqrt{x_1^2 + x_2^2 + x_3^2}$ has norm one. Second, the uniformity of $\lambda$ will follow if we prove that

$$\int_{\mathbb{R}^4} e^{-|y_1-x|^2} \, d\lambda(x) = \int_{\mathbb{R}^4} e^{-|y_2-x|^2} \, d\lambda(x) \quad \text{for any } y_1, y_2 \in \Lambda, \ r > 0. \quad (2)$$

We will prove that (2) leads to the uniformity at the end of the note. Since $\lambda$ is homogeneous, it suffices to prove the assertion (2) for the case $r = 1$ only. Using the rotational symmetry of $\Lambda$, we reduce (2) to the case where $y_1 = cy_2$, where $c > 0$ is a constant. In other words (see formula (1)),

$$\iint_{\mathbb{R}^3} \left( e^{-(x_1-t)^2-x_2^2-x_3^2} - e^{-(x_1-t)^2-x_2^2-x_3^2} - e^{-(x_1+t)^2-x_2^2-x_3^2} - e^{-(x_1+t)^2-x_2^2-x_3^2} \right) \, dx_1 \, dx_2 \, dx_3 \quad (3)$$

should be independent of $t > 0$. We are going to make the polar change of variables in this integral. Let us compute the part of the integral that does not depend on the radius $\sqrt{x_1^2 + x_2^2 + x_3^2}$ directly (this is, in fact, the computation of the Fourier transform of the Lebesgue measure on the sphere):

$$J_t(r) = \iint_{x_1^2 + x_2^2 + x_3^2 = r^2} e^{2x_1t} \, dx_1 \, dx_2 \, dx_3 = \sum_{1 \leq x_1^2 + x_3^2 \leq r^2} \iint_{x_1^2 + x_3^2 \leq r^2} e^{2r^2 - x_1^2} \frac{r \, dx_2 \, dx_3}{\sqrt{r^2 - x_1^2 - x_3^2}} =$$

$$\iint_{x_1^2 + x_3^2 \leq r^2} 2 \cosh \left( 2t \sqrt{r^2 - x_1^2 - x_3^2} \right) \frac{r \, dx_2 \, dx_3}{\sqrt{r^2 - x_1^2 - x_3^2}} = 2\pi \int_{0}^{r} 2 \cosh \left( 2t \sqrt{r^2 - r^2} \right) \frac{r \, dp}{\sqrt{r^2 - p^2}} \quad s = \sqrt{r^2 - p^2} =$$

$$-2\pi \int_{r}^{0} 2r \cosh(2ts) \, ds = -4\pi r \left[ \frac{\sinh(2ts)}{2t} \right]_{s=0}^{s=r} = \frac{2\pi r \sinh(2tr)}{t}.$$
We return to the expression in formula (3) and use the Fubini Theorem (we do not need to multiply by the Jacobian since \(J_t(r)\) is an integral over a 3-sphere of radius \(r\), not 1 like in the classical polar change of variables):

\[
\sum_{\pm} \int_{\mathbb{R}^3} e^{-(x_1-t)^2-x_2^2-x_3^2-(\sqrt{x_1^2+x_2^2+x_3^2} \pm t)^2} \, dx_1 \, dx_2 \, dx_3 = \\
\sum_{\pm} \int_{0}^{\infty} e^{-2r^2 \pm 2t^2} J_t(r) \, dr = 2\pi \sum_{\pm} \int_{0}^{\infty} e^{-2r^2 \pm 2t^2} \sinh(2tr) t^r \, dr.
\]

We have reduced the three-dimensional integral in (3) to a one-dimensional. We finish the computation:

\[
2\pi \sum_{\pm} \int_{0}^{\infty} e^{-2r^2 \pm 2t^2} \sinh(2tr) t^r \, dr = \\
\pi \int_{0}^{\infty} \left( e^{-2r^2+4rt-2t^2} - e^{-2r^2-4rt-2t^2} \right) r \, dr = \\
\pi \left( \int_{0}^{\infty} e^{-2(r-t)^2} t^r \, dr - \int_{0}^{\infty} e^{-2(r+t)^2} t^r \, dr \right) = \\
\pi \left( 2t - \frac{\pi}{2t} + \pi \int_{-\infty}^{0} e^{-2(r-t)^2} \, dr + \pi \int_{0}^{\infty} e^{-2(r-t)^2} \, dr \right) = \pi \int_{\mathbb{R}} e^{-2r^2} \, dr = \pi \sqrt{\frac{\pi}{2}}.
\]

Thus, (2) is proved. It remains to explain why this assertion leads to uniformity. Informally, the quantities \(\lambda(B_r(x))\) are expressible in terms of the \(\int e^{-r|x|^2} \, d\lambda(x)\) since the Gaussians are dense in any reasonable space of functions on the half-line. Here how the rigorous proof goes. It suffices to find (finite) collections of numbers \(\{a_k\}_k\) and \(\{s_k\}_k\) such that

\[
\left| \lambda(B_r(x)) - \sum_{k} a_k e^{-s_k |y-x|^2} \, d\lambda(y) \right|
\]

is arbitrarily small (and the collections \(\{a_k\}_k\) and \(\{s_k\}_k\) depend neither on \(x\) nor on \(\lambda\)). We estimate this difference by a suitable weighted \(L_2\)-norm using the fact that \(\frac{d}{dr} \lambda(B_r(x)) \lesssim (1 + r)^3\) (this follows from (2) and dilation considerations):

\[
\left| \lambda(B_r(x)) - \sum_{k} a_k e^{-s_k |y-x|^2} \, d\lambda(y) \right| \lesssim \int_{0}^{\infty} \left( \chi_{[0,r]}(t) - \sum_{k} a_k e^{-s_k t^2} \right) \, d\lambda(B_t(x)) \lesssim \int_{0}^{\infty} \left( \chi_{[0,r]}(t) - \sum_{k} a_k e^{-s_k t^2} \right)^2 (1 + t)^{10} \, dt.
\]

Thus, it suffices to show that the set of functions \(\{e^{-st^2}\}_{s>0}\) is dense in the space \(L^2(\mathbb{R}_+, (1 + t)^{10})\) (the space of functions \(f: \mathbb{R}_+ \to \mathbb{R}\) such that \(\int f^2(t)(1 + t)^{10} \, dt < \infty\)). Assume the contrary, then there exists a non-zero function \(g\) in the dual space \(L^2(\mathbb{R}_+, (1 + t)^{-10})\) such that \(\int g(t) e^{-st^2} \, dt = 0\) for any \(s > 0\). Consequently, a non-zero tempered distribution \(h\) given by the formula

\[
h(p) = \begin{cases} 
0, & p < 0; \\
p^{-\frac{1}{2}} g(\sqrt{p}), & p > 0
\end{cases}
\]

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satisfies $\hat{h}(is) = 0$ whenever $s > 0$. The function $\hat{h}$ is analytic in the upper half-plane, so $\hat{h} = 0$ there. We also have $\hat{h}(\cdot, is) \to \hat{h}(\cdot, 0)$ as $s \to 0$ at least in $S'(\mathbb{R})$. Therefore, $\hat{h} = 0$ on the real axis as a distribution, which contradicts the assumption that $h$ is non-zero.

References